Mathematical Tripos Part II
Michaelmas 2012
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Integrable Systems – Lecture 9[1]

The N-soliton solution... We have $F(x) = \sum_{n=1}^{N} \beta_n \mathrm{e}^{-\chi_n x}$, the GLM equation is

$$K(x,y) + \sum_{n=1}^{N} \beta_n \mathrm{e}^{-\chi_n(x+y)} + \int_x^\infty K(x,z) \sum_{n=1}^{N} \beta_n \mathrm{e}^{-\chi_n(x+z)}\,\mathrm{d}z = 0$$

and we seek solutions of the form $K(x,y) = \sum_{n=1}^{N} K_n(x)\mathrm{e}^{-\chi_n y}$. Substituting in the GLM equation,

$$\sum_{k=1}^{N} K_n(x)\mathrm{e}^{-\chi_n y} + \sum_{n=1}^{N} \beta_n \mathrm{e}^{-\chi_n(x+y)} + \int_x^\infty \sum_{m=1}^{N} K_m(x)\mathrm{e}^{-\chi_m z} \sum_{n=1}^{N} \beta_n \mathrm{e}^{-\chi_n(x+z)}\,\mathrm{d}z = 0$$

$$\Rightarrow \sum_{n=1}^{N} [K_n(x) + \beta_n \mathrm{e}^{-\chi_n x}]\mathrm{e}^{-\chi_n y} + \sum_{n=1}^{N} \left[\beta_n \sum_{m=1}^{N} \frac{K_m(x)}{\chi_m + \chi_n} \mathrm{e}^{-(\chi_m+\chi_n)x}\right] \mathrm{e}^{-\chi_n y} = 0.$$

Since $\mathrm{e}^{-\chi_1 y}, \ldots, \mathrm{e}^{-\chi_N y}$ are linearly independent, we deduce

$$K_n(x) + \beta_n \mathrm{e}^{-\chi_n x} + \beta_n \sum_{m=1}^{N} \frac{K_m(x)}{\chi_m + \chi_n} \mathrm{e}^{-(\chi_n+\chi_m)x} = 0, \qquad n = 1, \ldots, N.$$

To rewrite this in a matrix/vector form, let $A_{n,m}(x) = \delta_{n,m} + \beta_n \mathrm{e}^{-(\chi_n+\chi_m)x}/(\chi_n + \chi_m)$, $n, m = 1, \ldots, N$, whence

$$A\boldsymbol{k} = \boldsymbol{b}, \qquad \text{where} \qquad \boldsymbol{k} = \begin{bmatrix} K_1(x) \\ \vdots \\ K_N(x) \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} \beta_1 \mathrm{e}^{-\chi_1 x} \\ \vdots \\ \beta_N \mathrm{e}^{-\chi_N x} \end{bmatrix}.$$

Therefore $\boldsymbol{k} = -A^{-1}\boldsymbol{b}$. Moreover, by easy differentiation, $A'_{m,n}(x) = -b_m \mathrm{e}^{-\chi_n x}$, therefore

$$K(x,x) = \sum_{m=1}^{N} \mathrm{e}^{-\chi_m x} K_m(x) = -\sum_{m=1}^{N}\sum_{n=1}^{N} \mathrm{e}^{-\chi_m x}(A^{-1})_{m,n} b_n = \sum_{m=1}^{N}\sum_{n=1}^{N} (A^{-1})_{m,n} A'_{n,m}$$
$$= \sum_{m=1}^{N} (A^{-1}A')_{m,m} = \operatorname{tr} A^{-1}A' = \frac{1}{\det A}\frac{\mathrm{d}\det A}{\mathrm{d}x} = \frac{\mathrm{d}\log\det A}{\mathrm{d}x}.$$

Finally we restore the t-dependence,

$$u(x,t) = -2\frac{\mathrm{d}K(x,x)}{\mathrm{d}x} = -2\frac{\mathrm{d}^2 \log\det A(x)}{\mathrm{d}x^2}.$$

The case $N = 2$ Set $\tau_k = \chi_k x - 4\chi_k^3 t$, $k = 1, 2$. Then

$$\det A = \left[1 + \frac{\beta_1(0)}{2\chi_1}\mathrm{e}^{-2\tau_1}\right]\left[1 + \frac{\beta_2(0)}{2\chi_2}\mathrm{e}^{-2\tau_2}\right] - \frac{\beta_1(0)\beta_2(0)}{(\chi_1+\chi_2)^2}\mathrm{e}^{-2(\tau_1+\tau_2)}.$$

Let $t \to -\infty$. Then if also $x \to -\infty$, we have $\det A \sim \mathrm{e}^{-2(\tau_1+\tau_2)}$, hence

$$\log\det A \sim \text{const} - 2(\tau_1 + \tau_2), \qquad \lim_{x,t\to-\infty} u = 0,$$

[1]Please email all corrections and suggestions to these notes to `A.Iserles@damtp.cam.ac.uk`. All handouts are available on the WWW at the URL `http://www.damtp.cam.ac.uk/user/na/PartII/Handouts.html`.

which we already know. Let us now allow x grow up from $-\infty$. Then first τ_1 will vanish at $x_1 = 4\chi_1^2 t$, then τ_2 at $x_2 = 4\chi_2^2 t$ ($\chi_1 > \chi_2 > 0$). At x_1 we have $\tau_2 = 4t\chi_2(\chi_1^2 - \chi_2^2) \ll 1$ (since $t \ll -1$), hence

$$\det A \sim \frac{\beta_2(0)}{2\chi_2} \mathrm{e}^{-2\tau_2} \left[1 + \frac{\beta_1(0)}{2\chi_1} \left(\frac{\chi_1 - \chi_2}{\chi_1 + \chi_2}\right)^2 \mathrm{e}^{-2\tau_1}\right].$$

Thus, differentiating twice $\log \det A$,

$$u(x,t) \sim -2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \log \left(1 + \frac{\beta_1(0)}{2\chi_1} \left(\frac{\chi_1 - \chi_2}{\chi_1 + \chi_2}\right)^2 \mathrm{e}^{-2\chi_1(x - 4\chi_1^2 t)}\right)$$

which is similar to the one-soliton solution with the phase

$$(\phi_1)_- = \frac{1}{2\chi_1} \log \left(\frac{\beta_1(0)}{2\chi_1} \left(\frac{\chi_1 - \chi_2}{\chi_1 + \chi_2}\right)^2\right).$$

Similar calculation at x_2, where $\tau_1 \gg 1$, yields $\det A \sim 1 + \beta_2(0)/(2\chi_2)\mathrm{e}^{-2\tau_2}$ and u again looks 'like' a one-soliton solution, with the phase

$$(\phi_2)_- = \frac{1}{2\chi_2} \log \frac{\beta_2(0)}{2\chi_2}.$$

Now we start increasing t. As it approaches zero, the two solitons coalesce and the exact behaviour depends on χ_1/χ_2.

Next we start the 'movie' from $t \gg 1$. Now, as $x \to \infty$, we have $\det A \sim 1$ and $u(x,t) \sim 0$. We decrease x until $\tau_1 = 0$ (and $\tau_2 \gg 1$) – the solution looks like one soliton with the phase

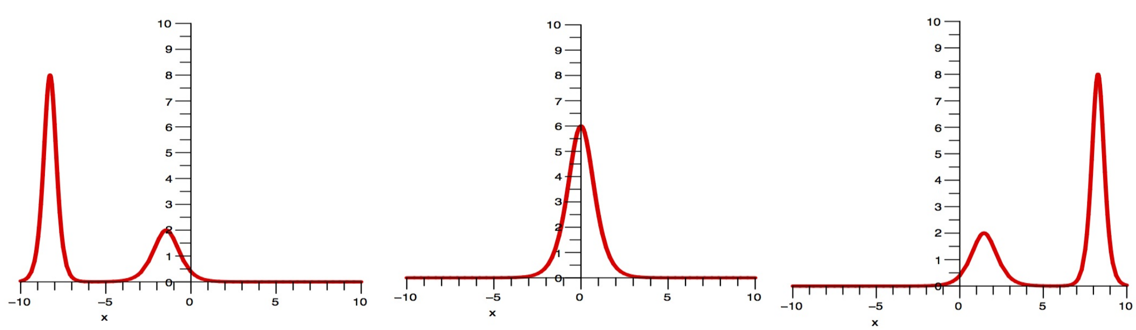
$$(\phi_1)_+ = \frac{1}{2\chi_1} \log \frac{\beta_1(0)}{2\chi_1}.$$

The x keeps decreasing, we reach $\tau_2 = 0$, $\tau_1 \ll -1$ and again we have a single soliton, with the phase

$$(\phi_2)_+ = \frac{1}{2\chi_2} \log \left(\frac{\beta_2(0)}{2\chi_2} \left(\frac{\chi_1 - \chi_2}{\chi_1 + \chi_2}\right)^2\right).$$

Thus, the larger soliton has overtaken the smaller one, they have preserved their shape *but their phases have changed,*

$$(\phi_1)_+ - (\phi_1)_- = -\frac{1}{\chi_1} \log \frac{\chi_1 - \chi_2}{\chi_1 + \chi_2}, \qquad (\phi_2)_+ - (\phi_2)_- = -\frac{1}{\chi_2} \log \frac{\chi_1 - \chi_2}{\chi_1 + \chi_2}.$$



The '2-soliton movie': for $t = -1$ (on the left) we have two solitons, a large on the left, a small on the right. At $t = 0$ they collide – note that the amplitude is *less* than the sum of the two amplitudes! At $t = 1$, on the right, the large solution has overtaken the small one, both have regained their shapes and aptitudes.