Back to the $N$-soliton solution  The ‘2-soliton movie’ generalises to any $N \geq 2$. Thus, the solution is made of $N$ separate solitons ordered according to their speed: the tallest (thus, fastest) is on the left, followed by next-tallest etc. At $t = 0$ they ‘interact’ and, for $t > 0$, individual solitons emerge in an opposite order. The total phase-shift is the sum of pairwise phase-shifts.

We know by now that the number $N$ of discrete Schrödinger eigenvalues equals the number of solitons for $t \neq 0$. The number is, of course, implicit in the initial conditions and this can be sometimes analysed further. For example, let

$$u(x, 0) = -\frac{N(N + 1)}{\cosh^2 x}, \quad N \in \mathbb{N}.$$  

Letting $\xi = \tanh x \in (-1, 1)$ in the Schrödinger equation $-f'' + u(x, 0)f = k^2 f$ yields the associated Legendre equation

$$\frac{d}{d\xi} \left[ (1 - \xi^2) \frac{df}{d\xi} \right] + \left[ N(N + 1) + \frac{k^2}{1 - \xi^2} \right] f = 0.$$  

Expanding $f$ into power series, it is possible to show that it is square-integrable in $(-1, 1)$ iff $k = in$ for $n \in \{1, 2, \ldots, N\}$. Therefore $\chi_n = N + 1 - n$.

### 4 Hamiltonian formalism of PDEs

First integrals  We claim that KdV has an infinity of first integrals. Recall that

$$\phi(x, k, t) = \begin{cases} e^{-ikx}, & x \rightarrow -\infty, \\ a(k, t)e^{-ikx} + b(k, t)e^{ikx}, & x \rightarrow \infty, \end{cases}  \quad (4.1)$$

where the KdV equation determines the time-dependence of the scattering data. Recall that $a(k, t) \equiv a(k, 0)$.

We will show that KdV has an infinity of first integrals, which we express in the form

$$I[u] = \int_{-\infty}^{\infty} P(u, u_x, u_{xx}, \ldots) \, dx,$$

where $P$ is a polynomial in its arguments. Set

$$\phi(x, k, t) = \exp \left( -ikx + \int_{-\infty}^{x} S(y, k, t) \, dy \right).$$

For $x \gg 1$ it follows from (4.1) that $e^{ikx}\phi(x, k, t) \approx a(k) + b(k, t)e^{2ikx}$. Letting $k \in \mathbb{C}^n$, the second term on the right-hand side tends to 0 as $x \to \infty$, thus

$$a(k) = \lim_{x \to \infty} e^{ikx}\phi(x, k, t) = \lim_{x \to \infty} \exp \left( \int_{-\infty}^{x} S(y, k, t) \, dy \right) = \exp \left( \int_{-\infty}^{\infty} S(y, k, t) \, dy \right).  \quad (4.2)$$

This remains true as $\text{Im} \, k \downarrow 0$ because of analyticity. Note that the integral above is independent of $t$!

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1Please email all corrections and suggestions to these notes to A.Iserles@damtp.cam.ac.uk. All handouts are available on the WWW at the URL http://www.damtp.cam.ac.uk/user/na/PartII/Handouts.html.
Thus, where $S$ such total derivatives generalises to all $S$


We seek asymptotic solution of the form $S(x, k, t) = \sum_{n=1}^{\infty} S_n(x, t)/(2ik)^n$ (which makes sense for sufficiently large $|k|$). Substitution into (4.3) gives

$$S_1(x, t) = -u(x, t), \quad S_{n+1}(x, t) = \frac{dS_n(x, t)}{dx} + \sum_{m=1}^{n-1} S_m(x, t)S_{n-m}(x, t).$$

The first few terms are

$$S_2 = -\frac{\partial u}{\partial x}, \quad S_3 = -\frac{\partial^2 u}{\partial x^2} + u^2, \quad S_4 = -\frac{\partial^3 u}{\partial x^3} + 2\frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2}, \quad S_5 = -\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^2 u}{\partial x^2} \left( \frac{\partial u}{\partial x} \right)^2 + 2u \frac{\partial^2 u}{\partial x^2} - 2u^3.$$ However... The $a(k)$s are independent of $t$, hence (4.2) implies that $\int_{-\infty}^{\infty} S_n(x, k, t) \, dx$ is a first integral of KdV.

Not all such integrals are non-trivial: actually, all the $S_{2n}$s are trivial! Thus,

$$\int_{-\infty}^{\infty} S_2(x, t) \, dx = -u(x, t) \bigg|_{x=\infty}^{x=-\infty} = 0, \quad \int_{-\infty}^{\infty} S_4(x, t) \, dx = -\frac{\partial^2 u(x, t)}{\partial x^2} + 2u^2(x, t) \bigg|_{x=\infty}^{x=-\infty} = 0$$

(we employ the boundary conditions and in the second expression we use the KdV to express $\partial^2 u/\partial x^2$ in terms of $u$ and $\partial u/\partial x$, which tend to 0 at $x \to \pm \infty$). The fact that both $S_2$ and $S_4$ are total derivatives generalises to all $S_{2n}$. To see this, consider $k \in \mathbb{R}$ and substitute $S = S_R + iS_I$, where $S_R$ and $S_I$ are real, into (4.3). The imaginary part is $dS_I/\, dx + 2S_RS_I - 2kS_R = 0$, therefore

$$S_R = -\frac{1}{2} \frac{dS_I}{dx} - \frac{1}{2} \frac{d}{dx} \log(S_I - k). \quad (4.4)$$

Because $k$ is real, the even terms $S_{2n}/(2ik)^{2n}$ are real for all $n \in \mathbb{N}$. Now, consider the asymptotic expansions of $S_R$ and $S_I$ separately: as we have just argued, $(S_R)_{2n} = S_{2n}$. On the other hand, it follows from (4.4) and the asymptotic expansion of $S_I$ that $(S_R)_{2n}$ is a total derivative. We deduce that

$$\int_{-\infty}^{\infty} S_{2n}(x) \, dx = 0$$

is trivial (i.e., doesn’t depend on $u$ being the solution of KdV) and should be disregard. Thus, we focus on the remaining integrals,

$$I_n[u] = \frac{1}{2} \int_{-\infty}^{\infty} S_{2n+3}(x, t) \, dx, \quad n \in \mathbb{Z}_+.$$ Thus,

$$I_0 = \frac{1}{2} \int_{-\infty}^{\infty} u^2(x, t) \, dx \quad \text{(energy)}, \quad I_1 = -\frac{1}{2} \int_{-\infty}^{\infty} [u_x^2(x, t) + 2u^3(x, t)] \, dx \quad \text{(momentum)}.$$ (Note that in the integrals above we discarded total derivatives, e.g. in $S_3$ the term $-d^2u/\, dx^2$.) These two integrals are associated through the Noether Theorem, with transitional invariance of KdV: if $u(x, t)$ is a solution then so are $u(x + x_0, t)$ and $u(x, t + t_0)$. A systematic investigation of such symmetries will be presented later in the course.