Mathematical Tripos Part II
Michaelmas 2012
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Integrable Systems – Lecture 10[1]

Back to the N-soliton solution The '2-soliton movie' generalises to any $N \geq 2$. Thus, the solution is made of N separate solitons ordered according to their speed: the tallest (thus, fastest) is on the left, followed by next-tallest etc. At $t = 0$ they 'interact' and, for $t > 0$, individual solitons emerge in an opposite order. The total phase-shift is the sum of pairwise phase-shifts.

We know by now that the number N of discrete Schrödinger eigenvalues equals the number of solitons for $t \neq 0$. The number is, of course, implicit in the initial conditions and this can be sometimes analysed further. For example, let

$$u(x,0) = -\frac{N(N+1)}{\cosh^2 x}, \qquad N \in \mathbb{N}.$$

Letting $\xi = \tanh x \in (-1,1)$ in the Schrödinger equation $-f'' + u(x,0)f = k^2 f$ yields the *associated Legendre equation*

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[(1-\xi^2)\frac{\mathrm{d}f}{\mathrm{d}\xi}\right] + \left[N(N+1) + \frac{k^2}{1-\xi^2}\right] f = 0.$$

Expanding f into power series, it is possible to show that it is square-integrable in $(-1,1)$ iff $k = \mathrm{i}n$ for $n \in \{1,2,\ldots,N\}$. Therefore $\chi_n = N + 1 - n$.

4 Hamiltonian formalism of PDEs

First integrals We claim that KdV has an infinity of first integrals. Recall that

$$\phi(x,k,t) = \begin{cases} \mathrm{e}^{-\mathrm{i}kx}, & x \to -\infty, \\ a(k,t)\mathrm{e}^{-\mathrm{i}kx} + b(k,t)\mathrm{e}^{\mathrm{i}kx}, & x \to \infty, \end{cases} \tag{4.1}$$

where the KdV equation determines the time-dependence of the scattering data. Recall that $a(k,t) \equiv a(k,0)$.

We will show that KdV has an infinity of first integrals, which we express in the form

$$I[u] = \int_{-\infty}^{\infty} P(u, u_x, u_{xx}, \ldots)\,\mathrm{d}x,$$

where P is a polynomial in its arguments. Set

$$\phi(x,k,t) = \exp\left(-\mathrm{i}kx + \int_{-\infty}^{x} S(y,k,t)\,\mathrm{d}y\right).$$

For $x \gg 1$ it follows from (4.1) that $\mathrm{e}^{\mathrm{i}kx}\phi(x,k,t) \approx a(k) + b(k,t)\mathrm{e}^{2\mathrm{i}kx}$. Letting $k \in \mathbb{C}^u$, the second term on the right-hand side tends to 0 as $x \to \infty$, thus

$$a(k) = \lim_{x\to\infty} \mathrm{e}^{\mathrm{i}kx}\phi(x,k,t) = \lim_{x\to\infty} \exp\left(\int_{-\infty}^{x} S(y,k,t)\,\mathrm{d}y\right) = \exp\left(\int_{-\infty}^{\infty} S(y,k,t)\,\mathrm{d}y\right). \tag{4.2}$$

This remains true as $\mathrm{Im}\,k \downarrow 0$ because of analyticity. Note that the integral above is independent of t!

[1]Please email all corrections and suggestions to these notes to A.Iserles@damtp.cam.ac.uk. All handouts are available on the WWW at the URL http://www.damtp.cam.ac.uk/user/na/PartII/Handouts.html.

We treat t as a parameter and use the Schrödinger equation, $-\phi'' + u\phi = k^2\phi$, to find S. Our *ansatz* for ϕ yields

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = [-\mathrm{i}k + S(x,k)]\phi, \qquad \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = \frac{\mathrm{d}S}{\mathrm{d}x}\phi + [-\mathrm{i}k + S(x,k)]^2\phi$$

and substitution in Schrödinger gives the Riccati equation

$$\frac{\mathrm{d}S}{\mathrm{d}x} - 2\mathrm{i}kS + S^2 = u. \tag{4.3}$$

We seek *asymptotic* solution of the form $S(x,k,t) = \sum_{n=1}^{\infty} S_n(x,t)/(2\mathrm{i}k)^n$ (which makes sense for sufficiently large $|k|$). Substitution into (4.3) gives

$$S_1(x,t) = -u(x,t), \qquad S_{n+1}(x,t) = \frac{\mathrm{d}S_n(x,t)}{\mathrm{d}x} + \sum_{m=1}^{n-1} S_m(x,t)S_{n-m}(x,t).$$

The first few terms are

$$S_2 = -\frac{\partial u}{\partial x}, \quad S_3 = -\frac{\partial^2 u}{\partial x^2} + u^2, \quad S_4 = -\frac{\partial^3 u}{\partial x^3} + 2\frac{\partial u^2}{\partial x}, \quad S_5 = -\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^2 u^2}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + 2u\frac{\partial^2 u}{\partial x^2} - 2u^3.$$

However... The $a(k)$s are independent of t, hence (4.2) implies that $\int_{-\infty}^{\infty} S_n(x,k,t)\,\mathrm{d}x$ is a first integral of KdV.

Not all such integrals are non-trivial: actually, all the S_{2n}s are trivial! Thus,

$$\int_{-\infty}^{\infty} S_2(x,t)\,\mathrm{d}x = -u(x,t)\Big|_{x=-\infty}^{x=\infty} = 0, \qquad \int_{-\infty}^{\infty} S_4(x,t)\,\mathrm{d}x = -\frac{\partial^2 u(x,t)}{\partial x^2} + 2u^2(x,t)\Big|_{x=-\infty}^{x=\infty} = 0$$

(we employ the boundary conditions and in the second expression we use the KdV to express $\partial^2 u/\partial x^2$ in terms of u and $\partial u/\partial x$, which tend to 0 at $x \to \pm\infty$). The fact that both S_2 and S_4 are total derivatives generalises to all S_{2n}. To see this, consider $k \in \mathbb{R}$ and substitute $S = S_R + \mathrm{i}S_I$, where S_R and S_I are real, into (4.3). The imaginary part is $\mathrm{d}S_I/\mathrm{d}x + 2S_RS_I - 2kS_R = 0$, therefore

$$S_R = -\tfrac{1}{2}\frac{\frac{\mathrm{d}S_I}{\mathrm{d}x}}{S_I - k} = -\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\log(S_I - k). \tag{4.4}$$

Because k is real, the even terms $S_{2n}/(2\mathrm{i}k)^{2n}$ are real for all $n \in \mathbb{N}$. Now, consider the asymptotic expansions of S_R and S_I separately: as we have just argued, $(S_R)_{2n} = S_{2n}$. On the other hand, it follows from (4.4) and the asymptotic expansion of S_I that $(S_R)_{2n}$ is a total derivative. We deduce that

$$\int_{-\infty}^{\infty} S_{2n}(x)\,\mathrm{d}x = 0$$

is trivial (i.e., doesn't depend on u being the solution of KdV) and should be disregarded. Thus, we focus on the remaining integrals,

$$I_n[u] = \tfrac{1}{2}\int_{-\infty}^{\infty} S_{2n+3}(x,t)\,\mathrm{d}x, \qquad n \in \mathbb{Z}_+.$$

Thus,

$$I_0 = \tfrac{1}{2}\int_{-\infty}^{\infty} u^2(x,t)\,\mathrm{d}x \quad \text{(energy)}, \qquad I_1 = -\tfrac{1}{2}\int_{-\infty}^{\infty} [u_x^2(x,t) + 2u^3(x,t)]\,\mathrm{d}x \quad \text{(momentum)}.$$

(Note that in the integrals above we discarded total derivatives, e.g. in S_3 the term $-\mathrm{d}^2u/\mathrm{d}x^2$.) These two integrals are associated through the *Noether Theorem*, with transitional invariance of KdV: if $u(x,t)$ is a solution then so are $u(x+x_0,t)$ and $u(x,t+t_0)$. A systematic investigation of such *symmetries* will be presented later in tho course.