Mathematical Tripos Part II
Michaelmas 2012
Professor A. Iserles

Integrable Systems – Lecture 11[1]

Hamiltonian formalism for PDEs We consider the Poisson formalism from Lecture 4 in a PDE setting. Thus, in place of variables $\xi^k(t)$, $k=1,\ldots,m$ we have the (real) function $u(x,t)$. Sums become integrals, functions acting on $\boldsymbol{\xi}$ become *functionals* acting on u and partial derivatives $\partial/\partial\xi^k$ become *functional derivatives* $\delta/\delta u$: for $F=\int_{-\infty}^{\infty} f(u,u_x,u_{xx},\ldots)\,\mathrm{d}x$ it is

$$\frac{\delta F[f]}{\delta u}=\frac{\partial f}{\partial u}-\frac{\partial}{\partial x}\frac{\partial f}{\partial u_x}+\frac{\partial^2}{\partial x^2}\frac{\partial f}{\partial u_{xx}}+\cdots,\qquad \frac{\delta u(y)}{\delta u(x)}=\delta(y-x).$$

In analogy with the ODE case, we define a Poisson bracket

$$\{F,G\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\omega(x,y,u)\frac{\delta F}{\delta u(x)}\frac{\delta G}{\delta u(y)}\,\mathrm{d}x\,\mathrm{d}y,$$

where the *Poisson structure* ω is such that the bracket obeys the axioms (it is skew-symmetric and the Jacobi identity holds). An important such structure is

$$\omega(x,y,u)=\tfrac12\frac{\partial\delta(x-y)}{\partial x}-\tfrac12\frac{\partial\delta(x-y)}{\partial y}.$$

(Derivatives of delta functions are computed integrating by parts.) With this *canonical* choice

$$\{F,G\}=\int_{\infty}^{\infty}\frac{\delta F}{\delta u}\frac{\partial}{\partial x}\frac{\delta G}{\delta u}\,\mathrm{d}x=-\int_{-\infty}^{\infty}\frac{\delta G}{\delta u}\frac{\partial}{\partial x}\frac{\delta F}{\delta u}\,\mathrm{d}x.$$

Thus, the Hamiltonian equations become

$$\frac{\partial u(x,t)}{\partial t}=\{u,H[u]\}=\int_{-\infty}^{\infty}\frac{\delta u(x)}{\delta u(y)}\frac{\partial}{\partial y}\frac{\delta H}{\delta u(y)}\,\mathrm{d}y=\frac{\partial}{\partial x}\frac{\delta H[u]}{\delta u}. \tag{4.5}$$

Over to KdV... Let $H[u]=-I_1[u]=\frac12\int_{-\infty}^{\infty}(u_x^2+2u^3)\,\mathrm{d}x$. Therefore

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\frac{\delta H[u]}{\delta u}=\tfrac12\frac{\partial}{\partial x}\left\{6u^2-2\frac{\partial}{\partial x}u_{xx}\right\}=6uu_x-u_{xxx}$$

and we recover the KdV equation. It is possible to show that, with our Poisson bracket, the integrals $\{I_n\}_{n\ge1}$ are in involution. For example, and bearing in mind the definition of I_n, using repeated integration by parts,

$$\begin{aligned}\{I_n,I_1\}&=\int_{-\infty}^{\infty}\frac{\delta I_n}{\delta u}\frac{\partial}{\partial x}\frac{\delta I_1}{\delta u}\,\mathrm{d}x=-\int_{-\infty}^{\infty}\frac{\delta I_n}{\delta u}\frac{\partial}{\partial x}\frac{\delta H}{\delta u}\,\mathrm{d}x=-\int_{-\infty}^{\infty}\frac{\delta I_n}{\delta u}u_t\,\mathrm{d}x\\&=-\tfrac12\int_{-\infty}^{\infty}\frac{\delta}{\delta u}\int_{-\infty}^{\infty}S_{2n+3}(\xi,t)\,\mathrm{d}\xi u_t\,\mathrm{d}x=-\tfrac12\int_{-\infty}^{\infty}\sum_{k=0}^{2n+2}(-1)^k\left[\frac{\partial^k}{\partial x^k}\frac{\partial S_{2n+3}}{\partial u^{(k)}}\right]u_t\,\mathrm{d}x\\&=\tfrac12\sum_{k=0}^{2n+2}\int_{-\infty}^{\infty}\frac{\partial S_{2n+3}}{\partial u^{(k)}}\frac{\partial u^{(k)}}{\partial t}\,\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}I_n[u]=0.\end{aligned}$$

Therefore, KdV is integrable in the Arnold–Liouville sense.

[1]Please email all corrections and suggestions to these notes to A.Iserles@damtp.cam.ac.uk. All handouts are available on the WWW at the URL http://www.damtp.cam.ac.uk/user/na/PartII/Handouts.html.

Bi-Hamiltonian systems Suppose that, for a given evolution equations, there exist two distinct Poisson structures $\mathcal{E}$ and $\mathcal{D}$ and two functionals $H_0[u]$ and $H_1[u]$ s.t.

$$\frac{\partial u}{\partial t} = \mathcal{D}\frac{\delta H_1[u]}{\delta u} = \mathcal{E}\frac{\delta H_0}{\delta u}.$$

For simplicity, let $\mathcal{D}$ be canonical, $\mathcal{D} = \partial/\partial x$, i.e. corresponding to (4.5).

Consider what it means having two Hamiltonians in an ODE setting. This corresponds to the existence of two skew-symmetric matrix functions ω and Ω that satisfy the Jacobi identity: we may assume that ω is constant, while $\Omega = \Omega(\boldsymbol{p}, \boldsymbol{q})$. Since Hamiltonian equations are $\xi_k' = \sum_{\ell=1}^{m} \omega_{\ell,k}(\boldsymbol{\xi})\partial H/\partial \xi_\ell$, we deduce that

$$\left(\omega_{\ell,k}\frac{\partial H_1}{\partial \xi_\ell}\right)_{k,\ell=1,\ldots,m} = \left(\Omega_{\ell,k}\frac{\partial H_0}{\partial \xi_\ell}\right)_{k,\ell=1,\ldots,m}.$$

Let $R_{k,j} = \sum_{\ell=1}^{m} \Omega_{\ell,j}(\omega^{-1})_{k,\ell}$, $k, j = 1, \ldots, m$. Then R is called a *recursion operator* and we can think of it as an endomorphism $\Omega \circ \omega^{-1}$ acting on the *tangent space* $T_p\mathcal{M}$, where $\boldsymbol{p} \in \mathcal{M}$. It is possible to prove that subject to further technical conditions, the existence of R is *equivalent* to Arnold–Liouville integrability. In particular, given that H_0 is an integral, we can construct integrals $H_1, \ldots, H_{n-1}$ recursively by

$$\left(\omega_{\ell,k}\frac{\partial H_i}{\partial \xi_\ell}\right)_{k,\ell=1,\ldots,m} = R^i \left(\omega_{\ell,k}\frac{\partial H_0}{\partial \xi_\ell}\right)_{k,\ell=1,\ldots,m}, \qquad i = 1, \ldots, n-1.$$

The formalism of recursion operators provides a convenient setting to extend Hamiltonian theory to PDEs, beyond the KdV equation. In the case of KdV we take

$$\mathcal{D} = \frac{\partial}{\partial x}, \quad H_1[u] = -I_1[u], \qquad \mathcal{E} = -\frac{\partial^3}{\partial x^3} + 4u\frac{\partial}{\partial x} + 2\frac{\partial}{\partial x}, \quad H_0[u] = I_0[u] = \tfrac{1}{2}\int_{-\infty}^{\infty} u^2(x,t)\,\mathrm{d}x.$$

Suppose that not just $\mathcal{D}$ and $\mathcal{E}$ are Poisson structures but so is $\mathcal{D}+c\mathcal{E}$, $c \in \mathbb{R}$. (The only problematic part is the Jacobi identity.) In that case the bi-Hamiltonian formulation provides a mechanism to construct a hierarchy of first integrals.

Theorem *Let us have a bi-Hamiltonian system where the Poisson structure $\mathcal{D}$ is non-degenerate. (We say that an operator $\mathcal{T}$ is degenerate if there exists an operator $\mathcal{S} \neq 0$ s.t. $\mathcal{S} \circ \mathcal{T} = 0$.) Let $R = \mathcal{E} \circ \mathcal{D}^{-1}$ be the corresponding recursion operator and assume that*

$$R^n\left(\mathcal{D}\frac{\delta H_0}{\delta u}\right) \in \operatorname{image}\mathcal{D}, \qquad n = 1, 2, \ldots.$$

Then there exist conserved functionals $\{H_m[u]\}_{m\in\mathbb{N}}$, which are in involution,

$$\{H_m, H_n\} = \int_{-\infty}^{\infty} \frac{\delta H_m}{\delta u}\mathcal{D}\frac{\delta H_n}{\delta u}\,\mathrm{d}x = 0, \qquad m, n \in \mathbb{N}.$$

We construct these functionals recursively,

$$\mathcal{D}\frac{\delta H_n}{\delta u} = R^n\left(\mathcal{D}\frac{\delta H_0}{\delta u}\right).$$

For KdV the recursion operator is $R = -\partial_x^2 + 4u + 2u_x\partial_x^{-1}$, where ∂_x^{-1} corresponds to integration. This gives an alternative means of constructing the first integrals I_n.