Mathematical Tripos Part II
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# Integrable Systems – Lecture 12[1]

**The zero-curvature representation** We generalise the setting of the Lax representation from KdV to more general integrable PDEs. Thus, let $U$ and $V$ be $n \times n$ matrix-valued functions of $(\rho, \tau, \lambda)$: there, $\rho$ and $\tau$ are standard PDE variables while $\lambda$ is the *spectral parameter.* Consider the coupled linear PDEs

$$\frac{\partial \boldsymbol{v}}{\partial \rho} = U(\lambda)\boldsymbol{v}, \qquad \frac{\partial \boldsymbol{v}}{\partial \tau} = V(\lambda)\boldsymbol{v}, \tag{4.6}$$

where $\boldsymbol{v} = \boldsymbol{v}(\rho, \tau, \lambda) \in \mathbb{C}^n$. (4.6) is an overdetermined system – $2n$ equations in $n$ unknowns – and we need to make it compatible. Since $\partial^2 \boldsymbol{v}/\partial\rho\partial\tau = \partial^2 \boldsymbol{v}/\partial\tau\partial\rho$, we differentiate the first equation w.r.t. $\tau$ and the second w.r.t. $\rho$, whereby

$$\boldsymbol{0} = \frac{\partial}{\partial \tau} U(\lambda)\boldsymbol{v} - \frac{\partial}{\partial \rho} V(\lambda)\boldsymbol{v} = \left( \frac{\partial U(\lambda)}{\partial \tau} - \frac{\partial V(\lambda)}{\partial \rho} + [U(\lambda), V(\lambda)] \right) \boldsymbol{v}.$$

This holds for all $\boldsymbol{v}$, therefore (4.6) is consistent iff

$$\frac{\partial U(\lambda)}{\partial \tau} - \frac{\partial V(\lambda)}{\partial \rho} + [U(\lambda), V(\lambda)] = 0. \tag{4.7}$$

This is known as the *zero-curvature representation* (the reason originates in differential geometry: the curvature of the connection $U\,\mathrm{d}\rho + V\,\mathrm{d}\tau$ is zero).

**Example 1** Let

$$U = \frac{\mathrm{i}}{2} \begin{bmatrix} 2\lambda & \phi_\rho \\ \rho_\rho & -2\lambda \end{bmatrix}, \qquad V = \frac{1}{4\mathrm{i}\lambda} \begin{bmatrix} \cos\phi & -\mathrm{i}\sin\phi \\ \mathrm{i}\sin\phi & -\cos\phi \end{bmatrix}.$$

Then (4.7) is the sine-Gordon equation $\phi_{\rho\tau} = \sin\phi$.

**Example 2** Let

$$U = \begin{bmatrix} \mathrm{i}\lambda & \mathrm{i}\bar{\phi} \\ \mathrm{i}\phi & -\mathrm{i}\lambda \end{bmatrix}, \qquad V = \begin{bmatrix} 2\mathrm{i}\lambda^2 - \mathrm{i}|\phi|^2 & 2\mathrm{i}\lambda\bar{\phi} + \bar{\phi}_\rho \\ 2\mathrm{i}\lambda\phi - \phi_\rho & -2\mathrm{i}\lambda^2 + \mathrm{i}|\phi|^2 \end{bmatrix}.$$

Then (4.7) yields the *nonlinear Schrödinger equation* $\mathrm{i}\phi_\tau + \phi_{\rho\rho} + 2|\phi|^2\phi = 0$.

The equation (4.7) admits freedom in the choice of $U, V$, known as the *gauge invariance.* Let $G = G(\tau, \rho)$ be an arbitrary invertible $n \times n$ matrix. Then

$$(U, V) \text{ soln of (4.7)} \quad \Rightarrow \quad (\tilde{U}, \tilde{V}) \text{ soln of (4.7), where } \tilde{U} = GUG^{-1} + G_\rho G^{-1},\ \tilde{V} = GVG^{-1} + G_\tau G^{-1}.$$

The proof, by direct substitution, is messy but elementary.

It is possible to extend inverse scattering to this setting. Suppose that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent solutions of (4.6) and let $\Phi = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]$. Then $\Phi$ is also a solution of (4.6) and we can set $U = \Phi_\rho \Phi^{-1}$, $V = \Phi_\tau \Phi^{-1}$. We'll consider briefly one general scheme for solving (4.7), the *dressing method,* but first few words on...

**The Riemann–Hilbert problem** Let $\lambda \in \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $\Gamma$ a closed contour in $\bar{\mathbb{C}}$. (In particular, $\Gamma$ can be the real line, a.k.a. the equator of the Riemann sphere.) The *Riemann–Hilbert*

[1]Please email all corrections and suggestions to these notes to `A.Iserles@damtp.cam.ac.uk`. All handouts are available on the WWW at the URL `http://www.damtp.cam.ac.uk/user/na/PartII/Handouts.html`.

*problem* is, given a matrix-valued function $G(\lambda)$, to construct two matrix functions, $G_+(\lambda)$ and $G_-(\lambda)$, holomorphic inside and outside $\Gamma$ respectively, s.t.

$$G(\lambda) = G_+(\lambda)G_-(\lambda), \qquad \lambda \in \Gamma. \tag{4.8}$$

(If $\Gamma$ is $\mathbb{R}$ then $G_+$ and $G_-$ are holomorphic in the upper and lower half planes resp.) Once $(G_+, G_-)$ is a solution to the RH problem then so is $(\tilde{G}_+, \tilde{G}_-)$, where $\tilde{G}_+ = G_+ g^{-1}$, $\tilde{G}_- = gG_-$ and $g$ is a constant nonsingular matrix. Ambiguity can be avoided e.g. by setting $G_-(\lambda_0) = I$ for some $\lambda_0 \in \bar{\mathbb{C}}$. If $G_+, G_-$ are invertible then, subject to the latter normalisation, the solution is unique.

Thus, let $G_+(\lambda_0) = I$ and set $g = G_-(\lambda_0)$ for some $\lambda_0$ outside $\Gamma$. We seek a solution of the RH problem of the form

$$G_+^{-1} = h + \oint_\Gamma \frac{\Phi(\xi)\,\mathrm{d}\xi}{\xi - \lambda}, \quad \lambda \text{ inside } \Gamma, \qquad G_- = h + \oint_\Gamma \frac{\Phi(\xi)\,\mathrm{d}\xi}{\xi - \lambda}, \quad \lambda \text{ outside } \Gamma,$$

where it follows from $G_-(\lambda_0) = g$ that $h = g - \oint_\Gamma \Phi(\xi)\,\mathrm{d}\xi/(\xi - \lambda_0)$. We now use the *Plemelj formula* to extend this to $\Gamma$:

$$G_+^{-1} = h + \oint \frac{\Phi(\xi)\,\mathrm{d}\xi}{\xi - \lambda} + \pi\mathrm{i}\Phi(\lambda), \quad G_- = h + \oint \frac{\Phi(\xi)\,\mathrm{d}\xi}{\xi - \lambda} - \pi\mathrm{i}\Phi(\lambda), \qquad \lambda \in \Gamma,$$

where we take principal values of integrals. Substitution into (4.8) yields an integral equation for $\Phi$. E.g. assuming that $h = g = 1$, it is

$$\frac{1}{\mathrm{i}\pi}\left[I + \int_\gamma \frac{\Phi(\xi)\,\mathrm{d}\xi}{\xi - \lambda}\right] + \Phi(G + I)(G - I)^{-1} = O.$$

In the simplest case of a *scalar* RH problem and $\Gamma = \mathbb{R}$, explicitly

$$G_+ = \exp\left(-\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \frac{\log G(\xi)\,\mathrm{d}\xi}{\xi - \lambda}\right), \quad \operatorname{Im}\lambda > 0, \qquad G_- = \exp\left(\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \frac{\log G(\xi)\,\mathrm{d}\xi}{\xi - \lambda}\right), \quad \operatorname{Im}\lambda < 0.$$

To verify, use the Cauchy integral formula on $\log G = \log G_- + \log G_+^{-1}$.

**The dressing method** Suppose that $U$ & $V$ in the zero-curvature representation (4.7) are rational functions of $\lambda$. Thus, they can be defined (up to multiplicative constants) by their poles and their multiplicities. (We can think about an $m$th degree polynomial as a pole at $\infty$ of multiplicity $m$.) Let the poles and their multiplicities be

$$\mathcal{S}_U = \{\alpha, \ldots, \alpha_N, \infty; n_1, \ldots, n_N, n_\infty\}, \qquad \mathcal{S}_V = \{\beta_1, \ldots, \beta_M, \beta_\infty; m_1, \ldots, m_M, m_\infty\}.$$

We say that $\mathcal{S}_U$ and $\mathcal{S}_V$ are the *divisors* of $U$ and $V$, resp. In other words,

$$U(\rho, \tau, \lambda) = \sum_{i=1}^{N}\sum_{\ell=1}^{n_i} \frac{U_{i,\ell}(\rho, \tau)}{(\lambda - \alpha_i)^\ell} + \sum_{k=0}^{n_\infty} U_k(\rho, \tau)\lambda^k,$$
$$V(\rho, \tau, \lambda) = \sum_{j=1}^{M}\sum_{\ell=1}^{m_j} \frac{V_{j,\ell}(\rho, \sigma)}{(\lambda - \beta_j)^\ell} + \sum_{k=0}^{m_\infty} V_k(\rho, \tau)\lambda^k.$$

In order to determine $U$ and $V$, we obtain from (4.7) a system of nonlinear PDEs for the coefficients $U_{i,\ell}, U_k, V_{j,\ell} V_k$.

To start with, let $U = U_0(\rho, \tau)$, $V = V_0(\rho, \tau)$ where $[U_0, V_0] = O$ (note that they are independent of $\lambda$) and have the divisors $\mathcal{S}_U$ and $\mathcal{S}_V$, respectively. It is trivial that $(U_0, V_0)$ is a solution of (4.7). The dressing method is a methodology to construct a nontrivial solution with the same divisors, commencing from $(U_0, V_0)$. (Hence the name: we commence from a "naked" solution and dress it up.)