Integrable Systems – Lecture 13[1]

. . . the dressing method Recall that (U, V) in the zero-curvature representation are rational in λ with divisors $\mathcal{S}_U$ & $\mathcal{S}_V$ resp.,

$$U(\rho,\tau,\lambda) = \sum_{i=1}^{N}\sum_{\ell=1}^{n_i} \frac{U_{i,\ell}(\rho,\tau)}{(\lambda-\alpha_i)^\ell} + \sum_{k=0}^{n_\infty} U_k(\rho,\tau)\lambda^k,$$
$$V(\rho,\tau,\lambda) = \sum_{j=1}^{M}\sum_{\ell=1}^{m_j} \frac{V_{j,\ell}(\rho,\sigma)}{(\lambda-\beta_j)^\ell} + \sum_{k=0}^{m_\infty} V_k(\rho,\tau)\lambda^k.$$

We let $U = U_0(\rho,\tau)$, $V = V_0(\rho,\tau)$ s.t. $[U_0, V_0] = O$. Choose a contour Γ in $\bar{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$ which stays clear of $\mathcal{S}_U \cup \mathcal{S}_V$ and let $G(\lambda)$ be a smooth matrix function defined for $\lambda \in \Gamma$. We wish to construct a solution to (4.7) with given divisors from the data $\{U_0, V_0, \Gamma, G\}$:

1. Find the fundamental solution of
$$\frac{\partial \Psi_0}{\partial \rho} = U_0(\lambda)\Psi_0, \qquad \frac{\partial \Psi_0}{\partial \tau} = V_0(\lambda)\Psi_0.$$
Note that, although the system is overdetermined, it is compatible because (U_0, V_0) obey (4.7).

2. Let
$$\tilde{G}(\rho,\tau,\lambda) = \Psi_0(\rho,\tau,\lambda)G(\lambda)\Psi_0^{-1}(\rho,\tau,\lambda)$$
be a family of smooth functions of $\lambda \in \Gamma$, parametrized by (ρ,τ). We factorize it by solving an RH problem:
$$\tilde{G}(\rho,\tau,\lambda) = G_+(\rho,\tau,\lambda)G_-(\rho,\tau,\lambda), \tag{4.9}$$
where G_+ and G_- are holomorphic inside and outside Γ, resp.

3. Differentiate (4.9) w.r.t. ρ (recall that G is independent of ρ):
$$\begin{aligned}\frac{\partial G_+}{\partial \rho}G_- + G_+\frac{\partial G_-}{\partial \rho} &= \frac{\partial \tilde{G}}{\partial \rho} = \frac{\partial}{\partial \rho}(\Psi_0 G\Psi^{-1}) = \frac{\partial \Psi_0}{\partial \rho}G\Psi_0^{-1} - \Psi_0 G\Psi_0^{-1}\frac{\partial \Psi_0}{\partial \rho}\Psi_0^{-1}\\ &= U_0\Psi_0 G\Psi_0^{-1} - \Psi_0 G\Psi_0^{-1}U_0\Psi_0\Psi_0^{-1} = U_0 G - GU_0 = U_0G_+G_- - G_+G_-U_0.\end{aligned}$$
We define
$$U(\rho,\tau,\lambda) = \left(\frac{\partial G_-}{\partial \rho} + G_-U_0\right)G_-^{-1} = -G_+^{-1}\left(\frac{\partial G_+}{\partial \rho} - U_0G_+\right).$$
(The equality follows from our analysis.) This function is analytic in $\bar{\mathbb{C}} \setminus \mathcal{S}_U$. According to the *Liouville Theorem* every analytic, bounded function in $\mathbb{C}$ is constant, this implies that U is rational in λ and that $\mathcal{S}_U = \mathcal{S}_{U_0}$. An identical argument results in
$$V(\rho,\tau,\lambda) = \left(\frac{\partial G_-}{\partial \tau} + G_-V_0\right)G_-^{-1} = -G_+^{-1}\left(\frac{\partial G_+}{\partial \tau} - V_0G_+\right)$$
and $\mathcal{S}_V = \mathcal{S}_{V_0}$.

[1]Please email all corrections and suggestions to these notes to A.Iserles@damtp.cam.ac.uk. All handouts are available on the WWW at the URL http://www.damtp.cam.ac.uk/user/na/PartII/Handouts.html.

4. Let
$$\Psi_+ = G_+^{-1}\Psi_0, \qquad \Psi_- = G_-^{-1}\Psi_0.$$
Then
$$\frac{\partial\Psi_+}{\partial\rho} = -G_+^{-1}\frac{\partial G_+}{\partial\rho}G_+^{-1}\Psi_0 + G_+^{-1}\frac{\partial\Psi_0}{\partial\rho} = -G_+^{-1}\frac{\partial G_+}{\partial\rho}G_+^{-1}\Psi_0 + G_+^{-1}U_0\Psi_0 = UG_+^{-1}\Psi_0 = U\Psi_+.$$
Likewise,
$$\frac{\partial\Psi_\pm}{\partial\rho} = U(\lambda)\Psi_\pm, \qquad \frac{\partial\Psi_\pm}{\partial\tau} = V(\lambda)\Psi_\pm.$$
Therefore (U,V) satisfy (4.7) (i.e., are a zero-curvature representation) and have prescribed divisors.

There is nothing to stop us from iterating this procedure. Thus, take another matrix function $\tilde{G}$ and repeat the procedure starting from (U,V), this will result in $\tilde{U},\tilde{V})$ with the same divisors. These *dressing transformations* form a group and
$$G = G_+G_-, \quad \tilde{G} = \tilde{G}_+\tilde{G}_-, \qquad \Rightarrow \qquad G\circ\tilde{G} = G_+\tilde{G}_+\tilde{G}_-G_-.$$
Solutions of RH problem, of course, are not unique, unless we fix the gauge. (Cf. the discussion from Lecture 12.) One usual such normalisation is by requiring that $G(\infty) = I$.
The application of the dressing method often requires additional constraints. For example, for sine-Gordon U & V need be *skew-Hermitian,* and this requires additional conditions on Γ and G.

Lax representation and zero curvature The Lax equation $\mathcal{L}_t = [\mathcal{A},\mathcal{L}]$ also arises as a compatibility condition for overdetermined PDEs. Thus, let $\mathcal{L}f = \lambda f$, λ simple. Recall that $0 = \lambda_t f = (\mathcal{L}-\lambda)(f_t + \mathcal{A}f)$, therefore $f_t + \mathcal{A}f = c(t)f$, where c is independent of x. We can thus use an integrating factor to find $\tilde{f} = \tilde{f}(x,t,\lambda)$ s.t. $\mathcal{L}\tilde{f} = \lambda\tilde{f}$, $\tilde{f}_t + \mathcal{A}\tilde{f} = 0$. In that case the Lax equation becomes a compatibility condition of an overdetermined system.

Let
$$\mathcal{L} = \sum_{k=0}^{n} u_k(x,t)\frac{\partial^k}{\partial x^k}, \qquad \mathcal{A} = \sum_{\ell=0}^{m} v_\ell(x,t)\frac{\partial^\ell}{\partial x^\ell}, \qquad u_n(x,t), v_m(x,t) \equiv 1.$$
Then the Lax eqns $\mathcal{L}_t = [\mathcal{A},\mathcal{L}]$ are nonlinear PDEs for the coefficients $\{u_0,\ldots,u_{n-1},v_0,\ldots,v_{m-1}\}$. The linear nth-order scalar eigenvalue problem $\mathcal{L}\tilde{f} = \lambda\tilde{f}$ is equivalent to the linear matrix PDE
$$\frac{\partial \boldsymbol{f}}{\partial x} = U_L\boldsymbol{f}, \qquad U_L = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \\ \lambda - u_0 & -u_1 & -u_2 & \cdots & -u_{n-2} & -u_{n-1} \end{bmatrix}, \quad \boldsymbol{f} = \begin{bmatrix} \tilde{f} \\ \frac{\partial\tilde{f}}{\partial x} \\ \vdots \\ \frac{\partial^{n-1}\tilde{f}}{\partial x^{n-1}} \end{bmatrix}.$$
Subject to the Lax eqn, $\tilde{f}_t + \mathcal{A}\tilde{f} = 0$. Differentiating repeatedly in x, we have
$$\partial_t\partial_x^k\tilde{f} + \sum_{\ell=0}^{k}\binom{k}{\ell}(\partial_x^{k-\ell}\mathcal{A})(\partial_x^\ell\tilde{f}) = 0.$$
We now use $\partial_x\boldsymbol{f} = U_L\boldsymbol{f}$, expressing $\partial_x^n\tilde{f}$ in terms of λ and lower derivatives, to rewrite $\partial_t\tilde{f}+\mathcal{A}\tilde{f} = 0$ in the form $\partial_t\boldsymbol{f} = V_A\boldsymbol{f}$. Cross-differentiating,
$$\frac{\partial\boldsymbol{f}}{\partial x} = U_L\boldsymbol{f}, \quad \frac{\partial\boldsymbol{f}}{\partial t} = V_A\boldsymbol{f} \qquad \Rightarrow \qquad \frac{\partial^2\boldsymbol{f}}{\partial x\partial t} = \frac{\partial U_L}{\partial t}\boldsymbol{f} + U_L\frac{\partial\boldsymbol{f}}{\partial t} = \frac{\partial V_A}{\partial x}\boldsymbol{f} + V_A\frac{\partial\boldsymbol{f}}{\partial x}$$
$$\Rightarrow \qquad \left(\frac{\partial U_L}{\partial t} - \frac{\partial V_A}{\partial x}\right)\boldsymbol{f} = (V_AU_L - U_LV_A)\boldsymbol{f} \qquad \Rightarrow \qquad \frac{\partial U_L}{\partial t} - \frac{\partial V_A}{\partial x} + [U_L,V_A] = 0$$
– we deduce that (U_L,V_A) obey zero compatibility conditions.