

## Integrable Systems – Lecture 14<sup>1</sup>

**Lax representation & zero curvature: an example** We apply the procedure to KdV: set  $f_0 = \tilde{f}(x, t, \lambda)$ ,  $f_1 = \partial_x \tilde{f}(x, t, \lambda)$ , then  $\mathcal{L}\tilde{f} = \lambda\tilde{f}$  yields  $\partial_x f_0 = f_1$ ,  $\partial_x f_1 = (u - \lambda)f_0$ . Next, we find from  $\partial_t \tilde{f} + \mathcal{A}\tilde{f} = 0$  that

$$\partial_t f_0 = -4\partial_x^3 f_0 + 6u f_1 + 3u_x f_0 = -u_x f_0 + 2(u + 2\lambda)f_1.$$

Differentiating w.r.t.  $x$  and eliminating the second derivatives of  $\tilde{f}$  yields

$$\partial_t f_1 = [2(u + 2\lambda)(u - \lambda) - u_{xx}]f_0 + u_x f_1.$$

We thus obtain a zero-curvature representation for KdV  $\partial_x \mathbf{f} = U_L \mathbf{f}$ ,  $\partial_t \mathbf{f} = V_A \mathbf{f}$ , where

$$\mathbf{f} = \begin{bmatrix} f_0 \\ f_1 \end{bmatrix}, \quad U_L = \begin{bmatrix} 0 & 1 \\ u - \lambda & 0 \end{bmatrix}, \quad U_A = \begin{bmatrix} -u_x & 2u + 4\lambda \\ 2u^2 + 2u\lambda - 4\lambda^2 - u_{xx} & u_x \end{bmatrix}.$$

## 5 Lie symmetries

**Lie groups and Lie algebras** A *Lie group* is a group which is also a manifold, such that the group operations (multiplication, inversion) are smooth w.r.t. the topology of the manifold. A group is a set  $\mathcal{G}$  with a map (multiplication)  $\mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$  which is associative ( $(g_1 g_2) g_3 = g_1 (g_2 g_3)$ ), has an identity ( $eg = g$ ) and an inverse  $gg^{-1} = g^{-1}g = e$ . Lie groups are important because they provide a powerful framework to discuss *symmetries* of differential equations.

Many finite-dimensional examples of Lie groups are *matrix groups*, e.g. the *general linear group*  $GL(n)$  of  $n \times n$  nonsingular matrices, the *special linear group*  $SL(n)$  of  $n \times n$  matrices with unit determinant and the *orthogonal group*  $O(n)$  of  $n \times n$  orthogonal matrices. However, not all finite-dimensional Lie groups are matrix groups.

Given a manifold  $\mathcal{M}$ , we say that  $\mathcal{G}$  acts on  $\mathcal{M}$  if there exists a map  $\mu : \mathcal{G} \times \mathcal{M} \rightarrow \mathcal{G}$  such that  $\mu(e, x) = x$ ,  $\mu(g_1, \mu(g_2, x)) = \mu(g_1 g_2, x)$ ,  $g_1, g_2 \in \mathcal{G}$ ,  $x \in \mathcal{M}$ . An example:  $\mathcal{M}$  an  $n$ -sphere,  $\mathcal{G} = O(n)$  (action by rotation).

Great insight into a Lie group is available by examining closely its elements near the identity. Let  $\lambda(t) = e + at + ct^2 + \dots$ ,  $\mu(t) = e + bt + dt^2 + \dots$  be two smooth function  $\mathbb{R} \rightarrow \mathcal{G}$  s.t.  $\lambda(0) = \mu(0) = e$ . What is the set  $\mathfrak{g}$  of all leading expansion terms  $a$  and  $b$  that can occur in this setting? Firstly, we note that  $\lambda(t)\mu(t) = e + (a + b)t + \mathcal{O}(t^2)$ , therefore  $a, b \in \mathfrak{g} \Rightarrow a + b \in \mathfrak{g}$  and  $\mathfrak{g}$  is a linear space. Moreover,  $\lambda^{-1}(t) = e - at + (a^2 - c)t^2 + \dots$ , hence  $\lambda^{-1}(t)\mu(t)\lambda(t)\mu^{-1}(t) = e + (ba - ab)t^2 + \dots$  – therefore  $a, b \in \mathfrak{g} \Rightarrow [a, b] \in \mathfrak{g}$ . Such a set  $\mathfrak{g}$  is called a *Lie algebra*. Formally, a Lie algebra  $\mathfrak{g}$  is a linear space closed under a binary, skew-symmetric operation  $[\cdot, \cdot]$  which satisfies the Jacobi identity  $[[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$ . What we have proved above is that *the tangent space to  $\mathcal{G}$  at  $e$  is endowed with a Lie-algebra structure*. Because  $\mathcal{G}$  is a group, once we know its tangent space at identity, we know it everywhere in  $\mathcal{G}$ .

The Lie algebra of  $GL(n)$  is  $\mathfrak{gl}(n)$ , the set of all  $n \times n$  matrices; the Lie algebra of  $SL(n)$  is  $\mathfrak{sl}(n)$ , the  $n \times n$  matrices with trace = 0 and the Lie algebra of  $O(n)$  is  $\mathfrak{so}(n)$ , the set of  $n \times n$  skew-symmetric matrices. [Check that all these sets are closed under commutation.] It is possible to prove that all finite-dimensional Lie algebras can be represented as matrix algebras (the Ado Theorem).

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<sup>1</sup>Please email all corrections and suggestions to these notes to [A.Iserles@damtp.cam.ac.uk](mailto:A.Iserles@damtp.cam.ac.uk). All handouts are available on the WWW at the URL <http://www.damtp.cam.ac.uk/user/na/PartII/Handouts.html>.

Let  $\mathfrak{g}$  be an  $n$ -dimensional (as a linear space!) Lie algebra and  $\{A_1, \dots, A_n\}$  its basis. Therefore there exist  $c_{k,\ell}^j$ ,  $k, \ell, j = 1, \dots, n$  s.t.

$$[A_k, A_\ell] = \sum_{j=1}^n c_{k,\ell}^j A_j, \quad k, \ell = 1, \dots, n.$$

The numbers  $\{c_{k,\ell}^j\}$  are called the *structure constants* of  $\mathfrak{g}$ .

**An example:** The *Heisenberg group*  $T_3$  of  $3 \times 3$  upper-triangular matrices with ones along the diagonal. [Prove that this is a group!] It is easy to confirm that the underlying *Heisenberg algebra*  $\mathfrak{t}_3$  is composed of strictly upper-triangular matrices. Taking the basis

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we have  $[A_1, A_2] = A_3$ ,  $[A_1, A_3] = 0$ ,  $[A_2, A_3] = 0$ , Hence  $c_{1,2}^3 = 1$ ,  $c_{2,1}^3 = -1$  and otherwise  $c_{k,\ell}^j = 0$ .

**Vector fields and transformation groups** An alternative to representing Lie algebras by matrices is to represent them as vector fields. Let  $\mathcal{X} \subset \mathbb{R}^n$  be open with local coordinates  $x_1, \dots, x_n$  and  $\gamma : [0, 1] \rightarrow \mathcal{X}$  be a parametrized smooth curve,  $\gamma(t) = (x_1(t), \dots, x_n(t))$ . We denote by  $V|_p$  the *tangent vector* to  $\gamma$  at  $p \in \mathcal{X}$ ,  $V_k = \dot{x}_k|_p$ ,  $k = 1, \dots, n$ . The  $n$ -dimensional linear space of all tangent vectors at  $p$  is the *tangent space*  $T_p\mathcal{X}$  and the collection of all tangent spaces is the *tangent bundle*  $T\mathcal{X} = \cup_{p \in \mathcal{X}} T_p\mathcal{X}$ : it is a manifold of dimension  $2n$ , linear in its first  $n$  coordinates.

A *vector field*  $\mathbf{V}$  over  $\mathcal{X}$  is a smooth assignment of a tangent vector  $V|_p \in T_p\mathcal{X}$  to every  $p \in \mathcal{X}$ . Consider a smooth  $f : \mathcal{R} \rightarrow \mathbb{R}$ . Its rate of change is

$$\left. \frac{df(x(t))}{dt} \right|_{t=0} = \sum_{k=1}^n V_k \frac{df}{dx_k} = \mathbf{V}(f), \quad \text{where} \quad \mathbf{V} = V_1 \frac{\partial}{\partial x_1} + \dots + V_n \frac{\partial}{\partial x_n}.$$

Consider the differential equation  $\mathbf{x}' = \mathbf{V}(\mathbf{x})$ , where  $\mathbf{V}$  is a vector field. Provided that  $\mathbf{x}(0) \in \mathcal{X}$ , it follows that the derivative is at the tangent space, *therefore*  $\mathbf{x}(t)$  *evolves in*  $\mathcal{X}$ . The *flow* corresponding to this ODE is the map  $\tilde{\mathbf{x}}(t, \mathbf{x}_0)$  such that  $\mathbf{x}(p) = \mathbf{x}_0$  implies that  $\mathbf{x}(t) = \tilde{\mathbf{x}}(t, \mathbf{x}_0)$ . Note that

$$\tilde{\mathbf{x}}(t, \mathbf{x}_0) = \mathbf{x}_0 + t\mathbf{V}(\mathbf{x}_0) + \mathcal{O}(t^2)$$

for  $t$  sufficiently small, hence the vector field  $\mathbf{V}$  is called the *generator* of the flow.

An *invariant* of a flow is a function  $f : \mathcal{X} \rightarrow \mathbb{R}$  s.t.  $f(\mathbf{x}(t)) = \text{const}$ , equivalently  $\mathbf{V}(f) = 0$ .

**Example I** Letting  $\mathbf{V} = x \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ , acting in  $\mathbb{R}^2$ , we obtain  $x'_1 = x_1$ ,  $x'_2 \equiv 1$ , therefore  $x_1(t) = e^t x_1(0)$ ,  $x_2(t) = x_2(0) + t$ . This is an example of a *one-parameter group of transformations*, because

$$\tilde{\mathbf{x}}(s, \tilde{\mathbf{x}}(t, \mathbf{x}_0)) = \tilde{\mathbf{x}}(t + s, \mathbf{x}_0), \quad s, t \geq 0, \quad \mathbf{x}_0 \in \mathcal{X}.$$

**Example II**  $\text{SO}(n) = \text{O}(n) \cap \text{SL}(n)$  is the Lie group of  $n \times n$  orthogonal matrices with unit determinant. In particular,  $\text{SO}(2)$  is the group of rotations of  $\mathbb{R}^2$ ,  $\mathbf{x}^\top = (x_1 \cos t - x_2 \sin t, x_1 \sin t + x_2 \cos t)$ . It is generated by

$$\mathbf{V} = \left. \frac{\partial \tilde{x}_1}{\partial t} \right|_{t=0} \frac{\partial}{\partial x_1} + \left. \frac{\partial \tilde{x}_2}{\partial t} \right|_{t=0} \frac{\partial}{\partial x_2} = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}.$$

It is easy to verify that the Euclidean norm  $(x_1^2 + x_2^2)^{1/2}$  is an invariant of  $\mathbf{V}$ .