Mathematical Tripos Part II
Michaelmas 2012
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Integrable Systems – Lecture 15[1]

The Lie bracket Let $\boldsymbol{V},\boldsymbol{W}$ be two vector fields. Their *Lie bracket* $[\boldsymbol{V},\boldsymbol{W}]$ is defined via

$$[\boldsymbol{V},\boldsymbol{W}](f)=\boldsymbol{V}(\boldsymbol{W}f)-\boldsymbol{W}(\boldsymbol{V}f) \qquad \Rightarrow \qquad [\boldsymbol{V},\boldsymbol{W}]_k=\sum_{\ell=1}^{n}\left(V_\ell\frac{\partial W_k}{\partial x_\ell}-W_\ell\frac{\partial V_k}{\partial x_\ell}\right).$$

Geometrically, a Lie bracket is the infinitesimal commutator of two flows, i.e. the difference between acting first with the first flow, then with the second, and the other way around. Thus, let $\tilde{x}_V$ and $\tilde{x}_W$ be the flows corresponding to $\boldsymbol{V}$ and $\boldsymbol{W}$ resp. and $f:\boldsymbol{X}\to\mathbb{R}$. Let

$$F(t,s,x)=f(\tilde{x}_V(t,\tilde{x}_W(s,x)))-f(\tilde{x}_W(s,\tilde{x}_V(t,x))). \qquad \text{Then} \qquad \left.\frac{\partial^2F(t,s,x)}{\partial t\partial s}\right|_{t=s=0}=[\boldsymbol{V},\boldsymbol{W}]f.$$

Example I The Heisenberg group, again: a 3×3 matrix group acting on $\mathbb{R}^3$,

$$\boldsymbol{y}=\begin{bmatrix}1 & m_1 & m_3\\ 0 & 1 & m_2\\ 0 & 0 & 1\end{bmatrix}\boldsymbol{x}=\begin{bmatrix}x_1+m_1x_2+m_3x_3\\ x_2+m_2x_3\\ x_3\end{bmatrix}.$$

The vector fields are

$$V_k=\sum_{\ell=1}^{3}\frac{\partial y_\ell}{\partial m_k}\frac{\partial}{\partial x_\ell} \qquad \Rightarrow \qquad V_1=x_2\frac{\partial}{\partial x_1},\quad V_2=x_3\frac{\partial}{\partial x_2},\quad V_3=x_3\frac{\partial}{\partial x_1}.$$

The Lie brackets are $[V_1,V_2]=-V_3$, $[V_1,V_3]=[V_2,V_3]=0$ – comparison with commutators in the underlying Lie algebra shows that structure constants differ only by a sign: in general, the Lie algebras spanned by vector fields and by matrices are isomorphic.

Example II Parallel parking. Suppose that a driver of a car has two transformations at their disposal:

$$\textsc{Steer}=\frac{\partial}{\partial\phi}, \qquad \textsc{Drive}=\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}+\frac{1}{L}\tan\phi\frac{\partial}{\partial\theta},$$

where $L>0$ is a constant, (x,y) are the coordinates of the centre of the rear axle, θ is the direction of the car and ϕ the angle between it and the front wheels. These flows don't commute and

$$[\textsc{Steer},\textsc{Drive}]=\frac{1}{L\cos^2\phi}\frac{\partial}{\partial\theta}=\textsc{Rotate}$$

– thus, Rotate is the manoeuvre Steer, Drive, Steer back, Drive back. Good – but not good enough for parallel parking. For this we need

$$\textsc{Slide}=[\textsc{Drive},\textsc{Rotate}]=\frac{1}{L\cos^2\phi}\left(\sin\theta\frac{\partial}{\partial x}-\cos\theta\frac{\partial}{\partial y}\right).$$

Perhaps you don't parallel-park using Lie groups: robots do, and that's why they are so good at it!

Symmetries of differential equations Let $u(x,t)$ be a solution to a DE with the variables (x,t) and let

$$\boldsymbol{V}=\xi(x,t,u)\frac{\partial}{\partial x}+\tau(x,t,u)\frac{\partial}{\partial t}+\eta(x,t,u)\frac{\partial}{\partial u}$$

[1] Please email all corrections and suggestions to these notes to A.Iserles@damtp.cam.ac.uk. All handouts are available on the WWW at the URL http://www.damtp.cam.ac.uk/user/na/PartII/Handouts.html.

act on $\mathbb{R}\times\mathbb{R}^2$. It generates a one-parameter *group of transformations* $(\tilde{x}(x,t,u,\varepsilon),\tilde{t}(x,t,u,\varepsilon),\tilde{u}(x,t,u,\varepsilon))$, called the *symmetry group* of the DE, provided that $\tilde{u}$ obeys the underlying DE.

For example, let the DE be KdV and $\boldsymbol{V}=\partial/\partial t$, in which case $\tilde{x}=x$, $\tilde{t}=t+\varepsilon$, while $\tilde{u}=u$ obeys the KdV eqn. In a hand-waiving manner, all it means it is that if $u(x,t)$ is a solution of KdV, so is $u(x,t+\varepsilon)$. More formally
Definition Let $\mathcal{X}=\mathbb{R}^n\times\mathbb{R}$ be the space of independent and dependent variables of a DE. A one-parameter group of transformations $(\tilde{x}(\boldsymbol{x},u,\varepsilon),\tilde{\boldsymbol{u}}(\boldsymbol{x},u,\varepsilon))$ is called a *Lie point symmetry group* of a DE $F(u,u_{x_1},\ldots,u_{x_n},u_{x_1,x_1},\ldots)=0$ if its action transforms solutions of this DE into other solutions of this DE. This can be extended to multi-parameter groups of transformations. Lie symmetries are useful for a number of reasons:

1. ***Construction of new solutions from old.*** For example, the *Lorenz group*

$$(\tilde{x},\tilde{t})=\left(\frac{x-\varepsilon t}{\sqrt{1-\varepsilon^2}},\frac{t-\varepsilon x}{\sqrt{1-\varepsilon^2}}\right),\qquad |\varepsilon|<1,$$

is a symmetry group of sine-Gordon: Let $\phi''+\sin\phi=0$ and $\psi(x,t)=\phi((x-\varepsilon t)/\sqrt{1-\varepsilon^2})$. Then $\psi_{tt}-\psi_{xx}-\sin\psi=\varepsilon^2\phi''/(1-\varepsilon^2)-\phi''/(1-\varepsilon^2)-\sin\phi=0$ and ψ solves sine-Gordon. This 'Lorenz boost' turns a static kink into a moving one.

2. ***Order reduction.*** Consider the DE $u'=F(u/t)$. It admits the *scaling symmetry* $(t,u)\to(\mathrm{e}^{\varepsilon}t,\mathrm{e}^{\varepsilon}u)$, $\varepsilon\in\mathbb{R}$, generated by $\boldsymbol{V}=x\partial/\partial t+u\partial/\partial u$. Let $r=u/t$, $s=\log|t|$ be new coordinates, hence $\boldsymbol{V}(r)\equiv 0$, $\boldsymbol{V}(s)\equiv 1$. If $F(r)=r$ then the general solution is $r=\mathrm{const}$, otherwise $\mathrm{d}s/\mathrm{d}r=(F(r)-r)^{-1}$ and the general (implicit) solution of the DE is

$$\log|t|+c_1=\int^{u/t}\frac{\mathrm{d}r}{F(r)-r}\qquad |t|=c\exp\left(\int^{u/t}\frac{\mathrm{d}r}{F(r)-r}\right).$$

3. ***Finding special solutions that admit symmetry.*** Although we cannot expect finding *all* solutions of the DE, we can find important special classes. For example, consider KdV and the vector field $\boldsymbol{V}=\partial/\partial t+c\partial/\partial x$: the corresponding invariants are u and $\xi=x-ct$. Thus, we seek a KdV solution of the form $u(x,t)=f(\xi)$: substituting into the equation yields a third-order ODE which integrates to $\frac{1}{2}(f')^2=f^3+\frac{1}{2}cf^2+\alpha f+\beta$: $u_t=-cf'$, $u_x=f'$, $u_{xx}=f''\Rightarrow -cf'-6ff'+f'''=0\Rightarrow$

$$-cf-3f^2+f''=\alpha\quad\Rightarrow\quad -cff'-3f^2f'+f''f'=\alpha f'\quad\Rightarrow\quad -\frac{c}{2}f^2-f^3+\frac{1}{2}f'^2=\alpha f+\beta.$$

This can be solved implicitly in terms of an elliptic integral,

$$\sqrt{2}\xi=\int^{\xi}\frac{\mathrm{d}f}{\sqrt{f^3+\frac{1}{2}cf^2+\alpha f+\beta}}.$$

In particular, if $f,f'\overset{\xi\to\pm\infty}{\longrightarrow}0$ then $\alpha=\beta=0$, the equation can be inverted and we recover the 1-soliton solution.

4. ***Recovery of conserved quantities*** Sadly, this is outside the scope of this course, although, paradoxically, this is a important contact point with integrability! According to the *Noether Theorem,* each continuous symmetry corresponds to a quantity conserved in time (essentially, a first integral). If we have a symmetry group of dimension $m\geq 1$, this means m conserved quantitates!