Mathematical Tripos Part II
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Professor A. Iserles

Integrable Systems – Lecture 16[1]

Finding symmetries Some symmetries can be just guessed: e.g., if there is no explicit dependence on $\boldsymbol{x}$ in the equation then the *translation* $\boldsymbol{x} \to \boldsymbol{x}+\boldsymbol{c}$ is a symmetry. All translations form an Abelian group, generated by $\partial/\partial x_1, \ldots, \partial/\partial x_n$. For general $(u, \partial_{x_k} u, \partial_{x_k}\partial_{x_\ell} u, \ldots) = 0$ it is possible to substitute $\tilde{u} = u + \varepsilon\eta(\boldsymbol{x}, u) + \mathcal{O}(\varepsilon^2)$, $\tilde{\boldsymbol{x}} = \boldsymbol{x} + \varepsilon\boldsymbol{\xi}(\boldsymbol{x}, u) + \mathcal{O}(\varepsilon^2)$ and retain linear terms in ε. A more systematic is *prolongation* of the vector field. E.g., assume $F(u, u_x, u_{xx}, u_{xxx}) = 0$ (KdV!). The prolongation of the vector field

$$\boldsymbol{V} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}$$

is

$$\mathrm{pr}(\boldsymbol{V}) = \boldsymbol{V} + \eta^t\frac{\partial}{\partial u_t} + \eta^x\frac{\partial}{\partial u_x} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xxx}\frac{\partial}{\partial u_{xxx}},$$

where the functions $\eta^t, \eta^x, \eta^{xx}, \eta^{xxx}$ of (u, x, t) will be determined algorithmically in the next section. $\mathrm{pr}(\boldsymbol{V})$ generates a 1-parameter group on transformations on the 7D space $(x, t, u, u_t, u_x, u_{xx}, u_{xxx})$ (an example of a *jet space*, where $u_t, u_x, \ldots$ are regarded as independent coordinates). $\boldsymbol{V}$ is a *symmetry* of the PDE $F = 0$ if $\mathrm{pr}(\boldsymbol{V})(F)\big|_{F=0} = 0$. This results in a linear PDE system for (ξ, τ, η) whose solution gives the general symmetry of the PDE.

Constructing the prolongation Suppose for simplicity that we wish to determine the symmetries of the ODE

$$\frac{\mathrm{d}^N u}{\mathrm{d}x^N} = F\left(x, u, \frac{\mathrm{d}u}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}^{N-1}u}{\mathrm{d}x^{N-1}}\right) \qquad \text{and consider the vector field} \qquad \boldsymbol{V} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}.$$

The prolongation is $\mathrm{pr}(\boldsymbol{V}) = \boldsymbol{V} + \sum_{k=1}^N \eta^{[k]}\frac{\partial}{\partial u^{(k)}}$ and it yields the transformation group

$$\tilde{x} = x + \varepsilon\xi + \mathcal{O}(\varepsilon^2), \qquad \tilde{u} = u + \varepsilon\eta + \mathcal{O}(\varepsilon^2), \qquad \tilde{u}^{(k)} = u^{(k)} + \varepsilon\eta^{[k]} + \mathcal{O}(\varepsilon^2), \quad k = 1, \ldots, N.$$

Set $D_x = \frac{\partial}{\partial x} + \sum_{k=1}^N u^{(k)}\frac{\partial}{\partial u^{(k-1)}}$, then by the chain rule $\tilde{u}^{(k)} = \frac{\mathrm{d}\tilde{u}^{(k-1)}}{\mathrm{d}\tilde{x}} = \frac{D_x\tilde{u}^{(k-1)}}{D_x\tilde{x}}$. Therefore

$$\tilde{u}^{(1)} = \frac{D_x\tilde{u}}{D_x\tilde{x}} = \frac{\frac{\mathrm{d}u}{\mathrm{d}x} + \varepsilon D_x(\eta) + \cdots}{1 + \varepsilon D_x(\xi) + \cdots} = \frac{\mathrm{d}u}{\mathrm{d}x} + \varepsilon\left(D_x\eta - \frac{\mathrm{d}u}{\mathrm{d}x}D_x\xi\right) + \mathcal{O}(\varepsilon^2) \quad \Rightarrow \quad \eta^{[1]} = D_x\eta - \frac{\mathrm{d}u}{\mathrm{d}x}D_x\xi.$$

Likewise, recursively, $\tilde{u}^{(k)} = [u^{(k)} + \varepsilon D_x\eta^{[k-1]} + \cdots]/[1 + \varepsilon D_x\xi + \cdots]$ yields $\eta^{[k]} = D_x\eta^{[k-1]} - \frac{\mathrm{d}^k u}{\mathrm{d}x^k}D_x\xi$ for all k. All this is entirely analogous (but much more complex) for PDEs.

Example Consider the simple ODE $u'' = 0$: we need to compute the second prolongation $\mathrm{pr}(\boldsymbol{V}) = \xi\partial_x + \eta\partial_u + \eta^x\partial_{u_x} + \eta^{xx}\partial u_{xx}$. Using the formula above,

$$\eta^x = \eta_x + (\eta_u - xi_x)u_x - \xi_u u_x^2, \quad \eta^{xx} = \eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x + (\eta_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 + (\eta_u - 2\xi_x)u_{xx} - 3\xi_u u_x u_{xx}.$$

Substituting this and the ODE into $\mathrm{pr}(\boldsymbol{V})(F)\big|_{F=0} = 0$, we have $\eta^{xx} = 0$. Since u_x is arbitrary (initial condition!) the coefficients of u_x^i, $i = 1, 2, 3$, vanish: $\eta_{xx} = 0$, $2\eta_{xu} - \xi_{xx} = 0$, $\eta_{uu} - 2\xi_{xu} = 0$, $\xi_{uu} = 0$. The general solution is an 8-dim symmetry group $\xi(x, u) = \varepsilon_1 x^2 + \varepsilon_2 xu + \varepsilon_3 x + \varepsilon_4 u + \varepsilon_5$, $\eta(x, u) = \varepsilon_1 xu + \varepsilon_2 u^2 + \varepsilon_6 x + \varepsilon_7 u + \varepsilon_8$.

Painlevé equation We consider ODEs in $\mathbb{C}$, hence all variables are complex. Commence from the linear ODE $w^{(N)} + \sum_{k=0}^{N-1} p_k(z)w^{(k)} = 0$. If $p_0, \ldots, p_{N-1}$ are analytic at $z = z_0$ then z_0 is

[1]Please email all corrections and suggestions to these notes to `A.Iserles@damtp.cam.ac.uk`. All handouts are available on the WWW at the URL `http://www.damtp.cam.ac.uk/user/na/PartII/Handouts.html`.

regular and (for any initial data) $w(z) = \sum_m a_m(z-z_0)^m$. Thus, singularities of w can be located only at the singularities of the p_ks and are independent of initial conditions. This is not true for nonlinear equations.

Examples $w' + w^2 = 0 \Rightarrow w(z) = (z-z_0)^{-1}$, a pole at an arbitrary z_0: a *movable* singularity. Likewise, $w'+w^3 = 0 \Rightarrow w(z) = [2(z-z_0)]^{-1/2}$ (movable singularity, a branch point) and $w'+\mathrm{e}^w = 0 \Rightarrow w(z) = \log(z-z_0)$ (ibid).

The Painlevé property The ODE $w^{(N)} = F(z, w, w', \ldots, w^{(N-1)})$, where F is rational in $w, w', \ldots, w^{(N-1)}$, has the *Painlevé property* if all its movable singularities are poles. Such 2nd-order equations (up to rational change of variable) have been characterised by Gambier, Painlevé & Kovalevskaya into 50 types. 44 of those can be brought into linear ODEs, the remaining 6 are the *Painlevé equations,* e.g.

$$\text{Painlevé 1:}\quad w'' = 6w^2 + z, \qquad \text{Painlevé 2:}\quad w'' = 2w^3 + wz + \alpha,$$

$$\text{Painlevé 3:}\quad w'' = \frac{1}{w}w'^2 - \frac{1}{z}w' + \frac{1}{z}(\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w},$$

$$\text{Painlevé 4:}\quad w'' = \frac{1}{2w}w'^2 + \tfrac{3}{2}w^3 + 4zw^2 + 2(z^2-\alpha)w + \frac{\beta}{w},$$

$$\text{Painlevé 5:}\quad w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)w'^2 - \frac{1}{z}w' + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1},$$

$$\text{Painlevé 6:}\quad w'' = \tfrac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)w'^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)w' + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right),$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are arbitrary. In other words, Painlevé property $\Leftrightarrow$ either the ODE is linearisable or it is can be transformed into one of these 6 Painlevé equations.

The Painlevé test In general, the integrability of PDEs (or even ODEs) is an open problem. However, Ablowitz, Ramani & Segur have noticed that PDEs integrable by inverse scattering reduce (once we impose invariance w.r.t. some Lie symmetries) to ODEs with Painlevé property. This provides a test for *necessary* conditions for integrability.

Example I: Sine-Gordon $\phi_{\rho\tau} = \sin\phi$, with the Lie symmetry $(\tilde{\rho}, \tilde{\tau}) = (c\rho, c^{-1}\tau)$, $c \neq 0$. Group-invariant solutions are $\phi(\rho, \tau) = F(\rho\tau)$, where $z = \rho\tau$ is an invariant of symmetry. Letting $w(z) = \mathrm{e}^{\mathrm{i}F(z)}$ in the PDE,

$$w'' = \frac{1}{w}w'^2 - \frac{1}{z}w' + \frac{1}{2z}w^2 - \frac{1}{2z},$$

which is a special case of Painlevé 3.

Example II: The *modified KdV* $u_t - 6u^2u_x + u_{xxx} = 0$ has a Lie symmetry $(\tilde{u}, \tilde{x}, \tilde{t}) = (c^\alpha u, c^\beta x, c^\gamma t)$, $c \neq 0$, when $\beta = -\alpha$, $\gamma = -3\alpha$. The corresponding symmetry group is generated by $\boldsymbol{V} = u\partial/\partial u - x\partial/\partial x - 3t\partial/\partial t$ and has two independent invariants, $z = 3^{-1/3}xt^{-1/2}$, $w = 3^{1/3}ut^{1/3}$. The group-invariant solutions are thus $w = w(z)$, which yields $u(x,t) = (3t)^{-1/3}w(z)$. Substituting into the PDE we obtain a 3rd-order ODE,

$$w_{zzz} - 6w^2w_z - w - zw_z = 0$$

which, once integrated, gives Painlevé 2.

The general Painlevé test (best done with symbolic computer program) boils down to three steps. First find all Lie symmetries. Then construct the ODE characterising the group-invariant solutions. Finally, check whether the latter has the Painlevé property.