Numerical Analysis – Lecture 15[1]

4 The Poisson equation

Problem 4.1 *(Approximation of ∇^2)* Our goal is to solve the *Poisson equation*

$$\nabla^2 u = f \qquad \forall \quad (x,y) \in \Omega, \tag{4.1}$$

where Ω is an open connected domain of $\mathbb{R}^2$ with a Jordan boundary, specified together with the *Dirichlet boundary condition*

$$u(x,y) = \phi(x,y) \qquad \forall \quad (x,y) \in \partial\Omega. \tag{4.2}$$

(You may assume that $f \in \mathrm{C}[\Omega]$, $\phi \in \mathrm{C}^2[\Omega]$, but this can be relaxed by an approach outside the scope of this course.) To this end we impose on Ω a square grid with uniform spacing of $\Delta x > 0$ and replace (4.1) by a *finite-difference* formula. For simplicity, we require for the time being that $\partial\Omega$ 'fits' into the grid: if a grid point lies inside Ω then all its neighbours are in $\mathrm{cl}\,\Omega$. We will discuss briefly in the sequel grids that fail this condition.

Remark 4.2 Finite differences are neither the only nor, arguably, the best means of solving partial differential equations. Other methods abound: finite elements, boundary elements, spectral and pseudospectral methods, finite-volume methods, vorticity methods, particle methods, meshless methods, gas-lattice methods and, in the important special case of the Poisson equation (4.1), fast multipole methods. Yet, finite difference are the simplest and the only ones to feature in this lecture course.

Back to Problem 4.1 Since $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, we need to consider the approximation of second derivatives.

Proposition 4.3 *Let $g \in C^4[a,b]$ and $x \in (a+\Delta x, b-\Delta x)$. Then*

$$g''(x) = \frac{1}{(\Delta x)^2}[g(x-\Delta x) - 2g(x) + g(x+\Delta x)] + \mathcal{O}\big((\Delta x)^2\big)\,. \tag{4.3}$$

Proof Expanding into Taylor series,

$$\begin{aligned} g(x-\Delta x) &= g(x) - \Delta x g'(x) + \tfrac{1}{2}(\Delta x)^2 g''(x) - \tfrac{1}{6}(\Delta x)^3 g'''(x) + \mathcal{O}\big((\Delta x)^4\big) \\ g(x+\Delta x) &= g(x) + \Delta x g'(x) + \tfrac{1}{2}(\Delta x)^2 g''(x) + \tfrac{1}{6}(\Delta x)^3 g'''(x) + \mathcal{O}\big((\Delta x)^4\big) \end{aligned}$$

and (4.3) follows by adding the two, subtracting $2g(x)$ and dividing by $(\Delta x)^2$. $\square$

Corollary 4.4 *The approximation*

$$\nabla^2 u(x,y) \approx \frac{1}{(\Delta x)^2}[u(x-\Delta x,y) + u(x+\Delta x,y) + u(x,y-\Delta x) + u(x,y+\Delta x) - 4u(x,y)]$$

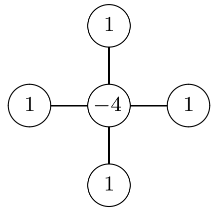
produces a (local) error of $\mathcal{O}\big((\Delta x)^2\big)$.

Approximation 4.5 The aforementioned analysis justifies the *five-point method*

$$u_{l-1,m} + u_{l+1,m} + u_{l,m-1} + u_{l,m+1} - 4u_{l,m} = (\Delta x)^2 f_{l,m}, \qquad (l\Delta x, m\Delta x) \in \Omega, \tag{4.4}$$

where $f_{l,m} = f(l\Delta x, m\Delta x)$, $u_{l,m} \approx u(l\Delta x, m\Delta x)$. It is usually denoted by the *computational stencil*

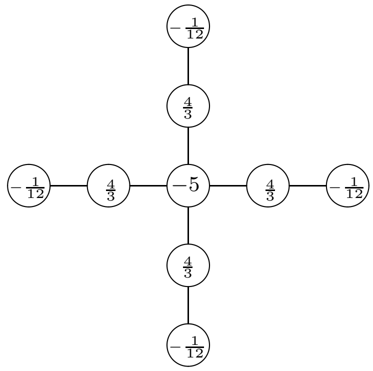
[1] Please email all corrections and suggestions to these notes to `A.Iserles@damtp.cam.ac.uk`. All handouts are available on the WWW at the URL `http://www.damtp.cam.ac.uk/user/na/PartII/Handouts.html`.



$$u_{l,m} = (\Delta x)^2 f_{l,m}$$

Whenever $(l\Delta x, m\Delta x) \in \partial\Omega$, we substitute appropriate Dirichlet boundary values. Note that the outcome of our procedure is a set of linear algebraic equations, whose solution approximates the solution of the Poisson equation (4.1) at the grid points.

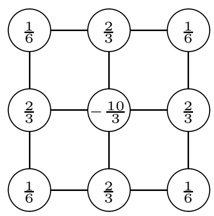
Approximation 4.6 It is easy (but laborious) to produce higher-order methods. You may verify, for example, that



$$u_{\ell,m} = (\Delta x)^2 f_{\ell,m}$$

produces a local error of $\mathcal{O}\big((\Delta x)^4\big)$. Needless to say, the implementation of this method is more complicated, since we might be 'missing' points near the boundary. Moreover, the set of algebraic equations that need be solved is less sparse than for the 5-point method, hence its solution is more expensive.

It is considerably easier to implement the *nine-point method*



$$u_{l,m} = (\Delta x)^2 f_{l,m}$$

but, unfortunately, it produces error of $\mathcal{O}\big((\Delta x)^2\big)$. This can be remedied by a clever trick which is outside the scope of this course.

Problem 4.7 *(Non-equispaced grids)* Since the boundary often fails to fit exactly into a square grid, we sometimes need to approximate ∇^2 using non-equispaced points. Clearly, it is enough to be able to approximate a second directional derivative w.r.t. each variable and subsequently 'synthetize' an approximation to ∇^2. For example, suppose that grid points are given with the spacing $\bullet\overset{\Delta x}{\text{———}}\bullet\overset{\kappa\Delta x}{\text{———}}\bullet$, where $0 < \kappa \le 1$. It is easy to verify by a Taylor expansion that

$$\frac{1}{(\Delta x)^2}\left[\tfrac{2}{\kappa+1}g(x-\Delta x) - \tfrac{2}{\kappa}g(x) + \tfrac{2}{\kappa(\kappa+1)}g(x+\kappa\Delta x)\right] = g''(x) + \tfrac{1}{2}(\kappa-1)g'''(x)\Delta x + \mathcal{O}\big((\Delta x)^2\big)\,,$$

with error of $\mathcal{O}((\Delta x))$ (note that $\kappa = 1$ gives, as expected, $\mathcal{O}\big((\Delta x)^2\big)$). Better approximation can be obtained by taking two equispaced points on the 'interior' side, i.e. $\bullet\overset{\Delta x}{\text{———}}\bullet\overset{\Delta x}{\text{———}}\bullet\overset{\kappa\Delta x}{\text{———}}\bullet$ as follows,

$$\begin{aligned}&\frac{1}{(\Delta x)^2}\left[\tfrac{\kappa-1}{\kappa+2}g(x-2\Delta x) - \tfrac{2(\kappa-2)}{\kappa+1}g(x-\Delta x) + \tfrac{\kappa-3}{\kappa}g(x) + \tfrac{6}{\kappa(\kappa+1)(\kappa+2)}g(x+\kappa\Delta x)\right]\\ &= g''(x) + \mathcal{O}\big((\Delta x)^2\big)\,.\end{aligned}$$