Numerical Analysis – Lecture 16[1]

Problem 4.8 Finite-difference discretization of $\nabla^2 u = f$ 'replaces' the PDE by a large system of linear equations. In the sequel we pay special attention to the *five-point formula,* which we rewrite in the form

$$-u_{l-1,j} - u_{l+1,j} - u_{l,j-1} - u_{l,j+1} + 4u_{l,j} = -(\Delta x)^2 f_{l,j}, \tag{4.5}$$

where $(l\Delta x, j\Delta x) \in \Omega$. Having ordered grid points, we can write (4.5) as a linear system, $A\boldsymbol{u} = \boldsymbol{b}$, say. (The precise manner of ordering grid points into a long vector is immaterial to our present discussion. The most obvious is *natural ordering,* by columns, but other orderings are of interest.) Recall from `Lecture 1` that if Ω is a square then A is symmetric and positive-definite.

Our present concern is to prove that, as $\Delta x \to 0$, the numerical solution (4.5) tends to the exact solution of the Poisson equation $\nabla^2 u = f$ (with appropriate Dirichlet boundary conditions). For the sake of simplicity, we restrict our attention to the important case of Ω being a *unit square,* whence $l, j = 1, 2, \ldots, m$, where $\Delta x = 1/(m+1)$.

Proposition 4.9 *The eigenvalues of the matrix A are*

$$\lambda_{\alpha,\beta} = 4\left\{\sin^2\left[\frac{\alpha\pi}{2(m+1)}\right] + \sin^2\left[\frac{\beta\pi}{2(m+1)}\right]\right\}, \qquad \alpha, \beta = 1, 2, \ldots, m.$$

Proof It is enough, given $\alpha, \beta \in \{1, 2, \ldots, m\}$, to demonstrate the existence of a nonzero vector $(v_{l,j})_{l,j=0}^{m+1}$ such that $v_{l,0} = v_{l,m+1} = v_{0,j} = v_{m+1,j} = 0$ for $l, j = 0, \ldots, m+1$ and

$$-v_{l-1,j} - v_{l+1,j} - v_{l,j-1} - v_{l,j+1} + 4v_{l,j} = \lambda_{\alpha,\beta} v_{l,j}, \qquad l, j = 1, 2, \ldots, m+1.$$

We let

$$v_{l,j} = \sin\left(\frac{l\alpha\pi}{m+1}\right)\sin\left(\frac{j\beta\pi}{m+1}\right), \qquad l, j = 0, 1, \ldots, m+1.$$

Note that the 'boundary conditions' are satisfied and, by virtue of the identity

$$\sin(\theta - \psi) + \sin(\theta + \psi) = 2\sin\theta\cos\psi.$$

We have

$$\begin{aligned}
-v_{l-1,j} - v_{l+1,j} - v_{l,j-1} - v_{l,j+1} + 4v_{l,j} &= -\left\{\sin\left[\frac{(l-1)\alpha\pi}{m+1}\right] + \sin\left[\frac{(l+1)\alpha\pi}{m+1}\right]\right\}\sin\left(\frac{j\beta\pi}{m+1}\right) \\
&\quad - \sin\left(\frac{l\alpha\pi}{m+1}\right)\left\{\sin\left[\frac{(j-1)\beta\pi}{m+1}\right] + \sin\left[\frac{(j+1)\beta\pi}{m+1}\right]\right\} \\
&\quad + 4\sin\left(\frac{l\alpha\pi}{m+1}\right)\sin\left(\frac{j\beta\pi}{m+1}\right) = \lambda_{\alpha,\beta} v_{l,j}.
\end{aligned}$$

This proves the proposition. □

As a matter of independent mathematical interest, note that for $1 \le \alpha, \beta \ll m$ we have

$$\frac{\lambda_{\alpha,\beta}}{(\Delta x)^2} \approx \frac{4}{(\Delta x)^2}\left[\frac{\alpha^2\pi^2}{4(m+1)^2} + \frac{\beta^2\pi^2}{4(m+1)^2}\right] = (\alpha^2 + \beta^2)\pi^2$$

[1] Please email all corrections and suggestions to these notes to `A.Iserles@damtp.cam.ac.uk`. All handouts are available on the WWW at the URL `http://www.damtp.cam.ac.uk/user/na/PartII/Handouts.html`.

and recall (e.g. from the solution of the Poisson equation in a square by separation of variables in `Maths Methods`) that the *exact* eigenvalues of $-\nabla^2$ are $(\alpha^2+\beta^2)\pi^2$, $\alpha, \beta \in \mathbb{N}$.

Let $\tilde{u}_{l,j} = u(l\Delta x, j\Delta x)$ (the exact solution of the Poisson equation) and $e_{l,j} = u_{l,j} - \tilde{u}_{l,j}$ (the pointwise error of the five-point formula). Set $\boldsymbol{e} = (e_{l,j})$.

Theorem 4.10 *Subject to sufficient smoothness of the function f and of the boundary conditions, there exists a number $c > 0$, independent of Δx, such that*

$$\|\boldsymbol{e}\| \le c\Delta x, \qquad \Delta x \to 0. \tag{4.6}$$

Proof We already know (having constructed the 5-point formula by matching Taylor expansions) that

$$-\tilde{u}_{l-1,j} - \tilde{u}_{l+1,j} - \tilde{u}_{l,j-1} - \tilde{u}_{l,j+1} + 4\tilde{u}_{l,j} = -(\Delta x)^2 f_{l,j} + \mathcal{O}\big((\Delta x)^4\big)\,.$$

Subtracting this from (4.5), we obtain

$$-e_{l-1,j} - e_{l+1,j} - e_{l,j-1} - e_{l,j+1} + 4e_{l,j} = \mathcal{O}\big((\Delta x)^4\big)\,. \tag{4.7}$$

Since $e_{l,j} = 0$ on the boundary, we deduce that, in a matrix notation, (4.7) can be written as $A\boldsymbol{e} = \boldsymbol{\delta}$, where $\|\boldsymbol{\delta}\| = \mathcal{O}\big((\Delta x)^3\big)$: the reason is that

$$\delta_{l,j} = \mathcal{O}\big((\Delta x)^4\big) \qquad \text{implies} \qquad \|\boldsymbol{\delta}\|^2 = \sum_{l=1}^{m}\sum_{j=1}^{m} \delta_{l,j}^2 = \mathcal{O}((\Delta x))^{-2} \times \mathcal{O}\big((\Delta x)^8\big)\,.$$

Therefore $\boldsymbol{e} = A^{-1}\boldsymbol{\delta}$.
The matrix A is symmetric, hence so is A^{-1} and it is true that $\|A^{-1}\| = \rho(A^{-1})$. The eigenvalues of A^{-1} can be deduced from Proposition 4.9, since they are the reciprocals of the eigenvalues of A. Thus, recalling that $\Delta x = 1/(m+1)$,

$$\begin{aligned}\rho(A^{-1}) &= \tfrac{1}{4} \max_{\alpha,\beta=1,\ldots,m} \left\{\sin^2\left[\frac{\alpha\pi}{2(m+1)}\right] + \sin^2\left[\frac{\beta\pi}{2(m+1)}\right]\right\}^{-1} \\ &= \frac{1}{8\sin^2(\frac{1}{2}\pi\Delta x)} \approx \frac{1}{2\pi^2(\Delta x)^2}.\end{aligned}$$

Therefore $\|\boldsymbol{e}\| \le \|A^{-1}\| \cdot \|\boldsymbol{\delta}\| \le c\Delta x$ for some constant $c > 0$. □

Problem 4.11 *(Solution of the 5-point equations)* We have already seen that

Jacobi: $\quad u_{l,j}^{(k+1)} = \frac{1}{4}\left[u_{l-1,j}^{(k)} + u_{l+1,j}^{(k)} + u_{l,j-1}^{(k)} + u_{l,j+1}^{(k)} - (\Delta x)^2 f_{l,j}\right];$

Gauss–Seidel: $\quad u_{l,j}^{(k+1)} = \frac{1}{4}\left[u_{l-1,j}^{(k+1)} + u_{l+1,j}^{(k)} + u_{l,j-1}^{(k+1)} + u_{l,j+1}^{(k)} - (\Delta x)^2 f_{l,j}\right].$

Moreover, it has been proved earlier in the lecture course (using Theorem 1.7) that both methods converge to the solution of (4.5). However, *the speed of convergence is very slow!* As a matter of fact, it is possible to prove that, denoting by B and $\mathcal{L}$ the iteration matrices of Jacobi and Gauss–Seidel respectively, it is true that (again, in a unit square)

$$\begin{aligned}\rho(B) &= \cos\left(\frac{\pi}{m+1}\right) \approx 1 - \frac{\pi^2}{2m^2}, \\ \rho(\mathcal{L}) &= \left[\cos\left(\frac{\pi}{m+1}\right)\right]^2 \approx 1 - \frac{\pi^2}{m^2}.\end{aligned}$$

Note that (at least asymptotically) Gauss–Seidel converges *at twice the speed of Jacobi.* Yet, even the speed of convergence of Gauss–Seidel is exceedingly slow. For example, $m = 100$ yields $\rho(B) \approx 0.9995$ and $\rho(\mathcal{L}) \approx 0.9990$. Requiring 6 significant digits, we need ≈ 27991 Jacobi iterations or ≈ 13996 iterations of Gauss–Seidel.