Numerical Analysis – Lecture 21[1]

Analysis 5.14 *(Fourier analysis of stability)* Let $\hat{u}^n = \sum_{m\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}m\theta} u_m^n$, $\theta \in [-\pi,\pi]$ be the Fourier transform of the sequence $\boldsymbol{u} \in \ell_2[\mathbb{Z}]$. We multiply the discretized equations

$$\sum_{k=-r}^{s} \alpha_k u_{m+k}^{n+1} = \sum_{k=-r}^{s} \beta_k u_{m+k}^{n}$$

by $\mathrm{e}^{-\mathrm{i}m\theta}$ and sum up for $m \in \mathbb{Z}$. Thus, the left-hand side yields

$$\begin{aligned}\sum_{m=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}m\theta} \sum_{k=-r}^{s} \alpha_k u_{m+k}^{n+1} &= \sum_{k=-r}^{s} \alpha_k \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}m\theta} u_{m+k}^{n+1} \\ = \sum_{k=-r}^{s} \alpha_k \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(m-k)\theta} u_m^{n+1} &= \left(\sum_{k=-r}^{s} \alpha_k \mathrm{e}^{\mathrm{i}k\theta} \right) \hat{u}^{n+1}(\theta).\end{aligned}$$

Similarly manipulating the right-hand side, we deduce that

$$\hat{u}^{n+1}(\theta) = H(\theta)\hat{u}^n(\theta) \quad \text{where} \quad H(\theta) = \frac{\sum_{k=-r}^{s} \beta_k \mathrm{e}^{\mathrm{i}k\theta}}{\sum_{k=-r}^{s} \alpha_k \mathrm{e}^{\mathrm{i}k\theta}}. \tag{5.4}$$

The function H is sometimes called the *amplification factor* of the recurrence (5.3).

Theorem 5.15 *The method (5.3) is stable iff $|H(\theta)| \le 1$ for all $\theta \in [-\pi,\pi]$.*

Proof The definition of stability is equivalent to the statement that there exists $C > 0$ such that $\|\boldsymbol{u}^n\| \le C\ \forall n \in \mathbb{Z}^+$. *[Because we are solving a Cauchy problem, equations are identical for all Δx, and this simplifies our analysis and eliminates a major difficulty: there is no need to insist explicitly that $\|\boldsymbol{u}^n\|$ remains uniformly bounded when $\Delta x \to 0$].* The Fourier transform being an isometry, stability is thus equivalent to $|||\hat{u}^n||| \le C\ \forall n \in \mathbb{Z}^+$. Iterating (5.4), however,

$$\hat{u}^n(\theta) = [H(\theta)]^n \hat{u}^0(\theta), \qquad |\theta| \le \pi, \quad n \in \mathbb{Z}^+. \tag{5.5}$$

Assume first that $|H(\theta)| \le 1\ \forall |\theta| \le \pi$. Then, by (5.5),

$$|\hat{u}^n(\theta)| \le |\hat{u}^0(\theta)| \quad \Rightarrow \quad |||\hat{u}^n|||^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\hat{u}^n(\theta)|^2 \,\mathrm{d}\theta \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |\hat{u}^0(\theta)|^2 \,\mathrm{d}\theta = |||\hat{u}^0|||^2.$$

Hence stability.
Suppose, on the other hand, that $\exists\theta_0 \in [-\pi,\pi]$ such that $|H(\theta_0)| = 1+\varepsilon > 1$, say. Since H is continuous, there exist $-\pi \le \theta_- < \theta_+ \le \pi$ such that $|H(\theta)| \ge 1 + \frac{1}{2}\varepsilon\ \forall\theta \in [\theta_-,\theta_+]$. We choose as our initial condition the function

$$\hat{u}^0(\theta) = \begin{cases} \sqrt{\frac{2\pi}{\theta_+ - \theta_-}}, & \theta_- \le \theta \le \theta_+, \\ 0, & \text{otherwise}. \end{cases}$$

[In case you insist on rendering an initial condition in the space of $\ell_2[\mathbb{Z}]$ sequences, equivalently choose

$$u_m^0 = \begin{cases} \sqrt{\frac{\theta_+ - \theta_-}{2\pi}}, & m = 0, \\ \frac{\mathrm{e}^{\mathrm{i}m\theta_+} - \mathrm{e}^{\mathrm{i}m\theta_-}}{m\sqrt{2\pi(\theta_+ - \theta_-)}}, & m \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

[1] Please email all corrections and suggestions to these notes to A.Iserles@damtp.cam.ac.uk. All handouts are available on the WWW at the URL http://www.damtp.cam.ac.uk/user/na/PartII/Handouts.html.

But we might just as well restrict our attention to the Fourier space.] Therefore

$$\||\hat{u}^n\|| = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\pi}^{\pi} |H(\theta)|^{2n} |\hat{u}^0(\theta)|^2 \, \mathrm{d}\theta \right\}^{\frac{1}{2}} = \frac{1}{\sqrt{2\pi}} \left\{ \int_{\theta_-}^{\theta_+} |H(\theta)|^{2n} |\hat{u}^0(\theta)|^2 \, \mathrm{d}\theta \right\}^{\frac{1}{2}}$$
$$\geq \frac{1}{\sqrt{2\pi}} \left(1 + \tfrac{1}{2}\varepsilon\right)^n \left\{ \int_{\theta_-}^{\theta_+} \frac{2\pi}{\theta_+ - \theta_-} \, \mathrm{d}\theta \right\}^{\frac{1}{2}} = \left(1 + \tfrac{1}{2}\varepsilon\right)^n \stackrel{n\to\infty}{\longrightarrow} \infty.$$

We deduce that the method is unstable. □

Example 5.16 Consider the Cauchy problem for the diffusion equation and recall the method

$$u_m^{n+1} = u_m^n + \mu(u_{m-1}^n - 2u_m^n + u_{m+1}^n).$$

Therefore we have

$$r = 1, \quad s = 1, \qquad \alpha_0 = 1, \quad \alpha_{\pm 1} = 0, \qquad \beta_0 = 1 - 2\mu, \quad \beta_{\pm 1} = \mu$$

in (5.3). We calculate

$$H(\theta) = 1 + \mu\left(\mathrm{e}^{-\mathrm{i}\theta} - 2 + \mathrm{e}^{\mathrm{i}\theta}\right) = 1 - 4\mu \sin^2 \tfrac{\theta}{2}, \qquad \theta \in [-\pi, \pi],$$

thus $1 \geq H(\theta) \geq H(\pi) = 1 - 4\mu$, and we deduce that the method is stable iff $\mu \leq \frac{1}{2}$. On the other hand, Crank–Nicolson, i.e.

$$u_m^{n+1} - \tfrac{1}{2}\mu(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}) = u_m^n + \tfrac{1}{2}\mu(u_{m-1}^n - 2u_m^n + u_{m+1}^n),$$

results in

$$H(\theta) = \frac{1 + \frac{1}{2}\mu(\mathrm{e}^{-\mathrm{i}\theta} - 2 + \mathrm{e}^{\mathrm{i}\theta})}{1 - \frac{1}{2}\mu(\mathrm{e}^{-\mathrm{i}\theta} - 2 + \mathrm{e}^{\mathrm{i}\theta})} = \frac{1 - 2\mu \sin^2 \frac{\theta}{2}}{1 + 2\mu \sin^2 \frac{\theta}{2}} \in (-1, 1] \qquad \theta \in [-\pi, \pi], \quad \mu > 0.$$

Hence stability for all $\mu > 0$.

Discussion 5.17 The difference between our former framework and Fourier analysis is that in the latter we stipulate that $m \in \mathbb{Z}$, which corresponds to $x \in \mathbb{R}$ in the original PDE. Thus, the above example is within a different framework to both the direct stability analysis of `Lecture 19` and to the eigenvalue analysis of `Lecture 20.` 'Translation' of Fourier analysis to problems with boundaries is by no means trivial and it often (although not for the diffusion equation, where we can get away with a much simpler argument!) calls for very deep functional-analytic tools.

It is frequently alleged that it is enough to 'pad' the vector u_m^n with zeros for $m \notin \{1, 2, \ldots, M-1\}$. This is in general wrong, since (e.g. when either $r \geq 2$ or $s \geq 2$ in (5.3)) we do not have enough boundary values to satisfy the equations near the boundary. This means that we must amend the discretized equations near the boundary and the identity (5.4) is no longer valid. In general, a great deal of care must be exercised to combine Fourier analysis with boundary conditions.

With many *parabolic* PDEs, e.g. the diffusion equation, the Euclidean norm of the exact solution decays (for zero boundary conditions) and good methods share this behaviour. Hence they are robust enough to cope with 'seepage' of energy from the boundary into the solution domain, which might occur when discretized equations are amended there. The situation is more difficult for many *hyperbolic equations*, e.g. the wave equation, since the exact solution keeps the energy (a.k.a. the Euclidean norm) constant and so do many good methods. In that case any 'seepage' of energy from the boundary can be dangerous, we must be careful and often we require further (heavy) mathematical machinery to deal with this situation.