Numerical Analysis – Lecture 22[1]

Problem 5.18 *(The diffusion equation in two space dimensions)* We are solving

$$\frac{\partial u}{\partial t} = \nabla^2 u, \qquad 0 \le x, y \le 1, \quad t \ge 0, \tag{5.6}$$

where $u = u(x, y, t)$, together with initial conditions at $t = 0$ and Dirichlet boundary conditions at $\partial[0,1]^2 \times [0, \infty)$. It is straightforward to generalize our derivation of numerical algorithms, e.g. by the method of lines. Thus, let $u_{l,m}(t) \approx u(l\Delta x, m\Delta x, t)$ and $u^n_{l,m} \approx u_{l,m}(n\Delta t)$. The five-point formula results in

$$u'_{l,m} = \frac{1}{(\Delta x)^2}(u_{l-1,m} + u_{l+1,m} + u_{l,m-1} + u_{l,m+1} - 4u_{l,m}).$$

Hence, Euler yields

$$u^{n+1}_{l,m} = u^n_{l,m} + \mu(u^n_{l-1,m} + u^n_{l+1,m} + u^n_{l,m-1} + u^n_{l,m+1} - 4u^n_{l,m}), \tag{5.7}$$

where, as before, $\mu = \Delta t/(\Delta x)^2$. The local error is $\mathcal{O}\big((\Delta x)^2\big)$. To analyse stability, we write (5.7) (with zero boundary conditions) in a vector form,

$$\boldsymbol{u}^{n+1} = A\boldsymbol{u}^n, \qquad n \in \mathbb{Z}^+.$$

It is easy to observe that A is an $M^2 \times M^2$ matrix, where $\Delta x = 1/(M+1)$, and that it is block-TST,

$$A = \begin{bmatrix} B & C & & \\ C & B & \ddots & \\ & \ddots & \ddots & C \\ & & C & B \end{bmatrix}, \text{ where } B = \begin{bmatrix} 1-4\mu & \mu & & \\ \mu & 1-4\mu & \ddots & \\ & \ddots & \ddots & \mu \\ & & \mu & 1-4\mu \end{bmatrix} \text{ and } C = \mu I.$$

Since both B and C are themselves TST, we deduce that A is symmetric, hence normal. Therefore, stability follows from $\rho(A) \le 1$.

The eigenvalues and eigenvectors of A can be explicitly written down: the proof is easy and it already featured (under a light disguise) in our presentation of Hockney's *fast Poisson solver.* To recap, the eigenvalues are

$$\lambda_{k,l} = 1 - 4\mu\left(\sin^2\frac{\pi k}{2M+2} + \sin^2\frac{\pi l}{2M+2}\right), \qquad k, l = 1, 2, \ldots, M,$$

the corresponding eigenvectors being $(\boldsymbol{v}_{k,l})_{i,j} = \sin\frac{\pi k i}{M+1}\sin\frac{\pi l j}{M+1}$, $i, j = 1, \ldots, M$. Consequently,

$$\sup_{\Delta x > 0}\rho(A) = \max\{1, |1 - 8\mu|\}$$

and $\mu \le \frac{1}{4} \Leftrightarrow$ stability.

Fourier analysis, likewise, generalizes to two dimensions: of course, we now need to extend the range of (x, y) in (5.6) from $0 \le x, y \le 1$ to $x, y \in \mathbb{R}$. A 2D Fourier transform reads

$$\hat{v}(\theta, \psi) = \sum_{l,m\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}(l\theta + m\psi)} v_{l,m}$$

[1] Please email all corrections and suggestions to these notes to `A.Iserles@damtp.cam.ac.uk`. All handouts are available on the WWW at the URL `http://www.damtp.cam.ac.uk/user/na/PartII/Handouts.html`.

and all our results readily generalize. In particular, the Fourier transform is an isometry from $\ell[\mathbb{Z}^2]$ to $L_2([-\pi,\pi]^2)$, i.e.

$$\left(\sum_{k,l=-\infty}^{\infty}|v_{k,l}|^2\right)^{1/2}=\|\boldsymbol{v}\|=|||\hat{v}|||=\left(\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\hat{v}(\theta,\psi)|^2\mathrm{d}\theta\mathrm{d}\psi\right)^{1/2}.$$

Moreover, (5.7) is stable iff $|H(\theta,\psi)|\le 1$, $-\pi\le\theta,\psi\le\pi$, where

$$H(\theta,\psi)=1+\mu(\mathrm{e}^{-\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\psi}+\mathrm{e}^{\mathrm{i}\psi}-4)=1-4\mu\left(\sin^2\tfrac{\theta}{2}+\sin^2\tfrac{\psi}{2}\right).$$

The proofs are an easy elaboration on the one-dimensional theory and are left to the reader. Insofar as the method (5.7) is concerned, we again deduce stability $\Leftrightarrow\mu\le\frac{1}{4}$.

Problem 5.19 *(The advection equation)* A useful paradigm for hyperbolic PDEs is the *advection equation*

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x},\qquad 0\le x\le 1,\quad t\ge 0,\tag{5.8}$$

where $u=u(x,t)$. It is given with the initial condition $u(x,0)=\varphi(x)$, $x\in[0,1]$ and (for simplicity) the boundary condition $u(1,t)=\varphi(t+1)$. The exact solution of (5.8) is simply $u(x,t)=\varphi(x+t)$, a unilateral shift leftwards. This, however, does not mean that its numerical modelling is easy…

We commence by discretizing $\partial u(m\Delta x,t)/\partial x\approx(\Delta x)^{-1}[u_{m+1}(t)-u_m(t)]$, $m=0,\ldots,M$, and solve the ODE by Euler's method. The outcome is

$$u_m^{n+1}=u_m^n+\mu(u_{m+1}^n-u_m^n),\qquad m=0,1,\ldots,M,\quad n\ge 0,\tag{5.9}$$

where the Courant number is $\mu=\Delta t/\Delta x$. It follows at once from our construction that the local error is $\mathcal{O}\big((\Delta x)^2\big)$.

To analyse stability, we let the boundary condition at $x=1$ be zero and, á la `Lecture 19` and define $\eta^n=\max_{m=0,\ldots,M}|u_m^n|$. It follows at once from (5.9) that

$$|u_m^{n+1}|\le|1-\mu||u_m^n|+\mu|u_{m+1}^n|\le(|1-\mu|+\mu)\eta^n,\qquad n\in\mathbb{Z}^+.$$

Therefore, $\mu\in(0,1]$ means that $\eta^{n+1}\le\eta^n\le\cdots\le\eta^0$, hence uniform boundedness, hence stability. *[Note that eigenvalue analysis of stability does not apply here, since the matrix is not normal! Indeed, this is an example of simplistic analysis of eigenvalues leading to a* wrong *answer! Fourier analysis is inapplicable either, since this is not a Cauchy problem, but it can be generalized easily to cater for (5.9).]*

Method 5.20 *(The leapfrog method)* We semidiscretize in equation (5.8) with the finite-difference approximation $\partial u(m\Delta x,t)/\partial x\approx\frac{1}{2}(\Delta x)^{-1}[u_{m+1}(t)-u_{m-1}(t)]$ and solve the ODE $\boldsymbol{y}'=\boldsymbol{f}(t,\boldsymbol{y})$ with the second-order *midpoint rule*

$$\boldsymbol{y}_{n+1}=\boldsymbol{y}_{n-1}+2h\boldsymbol{f}(t_n,\boldsymbol{y}_n),\qquad n=1,2,\ldots.$$

The outcome is the two-step *leapfrog* method

$$u_m^{n+1}=\mu(u_{m+1}^n-u_{m-1}^n)+u_m^{n-1}.\tag{5.10}$$

The error is now $\mathcal{O}\big((\Delta x)^3\big)$.

We analyse stability by the Fourier technique, assuming that we are solving a Cauchy problem. Thus, proceeding as before,

$$\hat{u}^{n+1}(\theta)=\mu(\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta})\hat{u}^n(\theta)+\hat{u}^{n-1}(\theta)=2\mathrm{i}\mu(\sin\theta)\hat{u}^n(\theta)+\hat{u}^{n-1}(\theta).\tag{5.11}$$

We defer to the next lecture a discussion whether the method (5.10) is stable: this is the same as $|\hat{u}^n(\theta)|\le c$ for all $|\theta|\le\pi$ and $n\in\mathbb{Z}^+$.