# further Examples

This document contains discussion for the examples from Curve Interrogation onward.

## Curve Interrogations

*Newton iteration*

Follow the Newton iteration seeking for a root of $0 = x - x^3$ starting from various initial values of $x$
a) between 0.1 and 0.5
b) between 1.1 and 1.5
c) at 0.9
d) between 0.5 and 0.71

What can you say about the different patterns of convergence ?

• The patterns are essentially that (a) gives points of oscillating sign converging on the root at $x = 0$, the result of the inflexion at the root, (b) converges on the root at $x = 1$ from one side only. This is the normal behaviour when the function has a non-zerfo curvature at the root. (c) has a single signn change and then approaches from one side as before. In (d) the behaviour is much more chaotic, with different starting points not differing by very much converging to different roots. In fact there are an infinite number of intervals, each maximal converging to a particular root, within that range.

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Now look for a root of $0 = 1 - x + x^2$ with initial values
e) 0
f) 0.1
g) 0.01

• These just oscillate around without ever finding a root. Not surprising, as there is no real root to converge to. (e) gives a stable cycle, (f) and (g) slowly diverge from it.

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## Parametric Surface Definitions

*Implicit Surfaces*

These are not parametric but it is worth having one question on them.

What are the surfaces defined by

$$f1(P) = x^2 + y^2 - 1$$
$$f2(P) = z$$
$$f3(P) = 4 - x^2 + y^2 + z^2$$

and the solid defined by

$$f4(P) = min(f1(P), f2(P), f3(P)) >= 0$$

• $f1 = 0$ is a cylinder of radius one, centred on the $z$-axis, $f2 = 0$ is the $xy$-plane, and $f3 = 0$ is a sphere of radius 2 centred at the origin.

The corresponding half-spaces are:- $f1 > 0$ is the exterior of the cylinder (i.e. all of space, with a cylindrical hole), $f2 > 0$ is the +ve $z$ half-space, and $f3 > 0$ is the interior of the sphere.

The min function in $f4$ is the equivalent of the intersection function, and so it defines a hemisphere with a hole drilled through it.

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*Tensor Product: Inheritance of various properties*

Prove that, if $f_i(u)$ is non-negative for all $i \in 1..m$ and all $u \in [0..1]$ and $g_j(v)$ is non-negative for all $j \in 1..n$ and all $v \in [0..1]$ , then $f_i(u)g_j(v)$ is non-negative for all $i \in 1..m$, all $j \in 1..n$ and all $(u,v) \in [0..1]^2$ .

- The product $f_i(u)g_j(v)$ is the product of two non-negative numbers, which cannot be negative. Don't let all the notation fool you into thinking this is complicated.

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Prove that, if $\sum_{i\in 1..m} f_i(u) = 1$ for all $u \in [0..1]$ and $\sum_{j\in 1..n} g_j(v) = 1$ for all $v \in [0..1]$, then

$$\sum_{i\in 1..m} \sum_{j\in 1..m} f_i(u)g_j(v) = 1 \forall (u,v) \in [0..1]^2$$

- $f_i(u)$ is constant over the inner summation, so we can rewrite the double sum as

$$\sum_{i\in 1..m} (f_i(u) \sum_{j\in 1..m} g_j(v)) \forall (u,v) \in [0..1]^2$$

The inner sum is now unity by axiom 2, so the double sum reduces to

$$\sum_{i\in 1..m} (f_i(u)) \forall (u,v) \in [0..1]^2$$

which in turn reduces to 1 by axiom 1.

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Prove that, if $f_i(u)$ is $Cn$ over $[0..1]$ and $g_j(v)$ is $Cn$ over $[0..1]$, then $f_i(u)g_j(v)$ is $Cn$ over $(u,v) \in [0..1]^2$

- This is not quite so trivial, because we have to consider continuity along a path which crosses a discontinuity in $f_i(u)$ and one in $g_j(v)$ at the same point.

For simplicity, take such a point as origin in the $uv$-plane (the transformation $u' = u - u_0$, $v' = v - v_0$ achieves this), and let

$$f_i(u) = \left( \begin{array}{cc} f(u) & u < 0 \\ f(u) + \alpha u^{n+1} + O(u^n + 2) & u \geq 0 \end{array} \right.$$

and

$$g_j(u) = \left( \begin{array}{cc} g(v) & v < 0 \\ g(v) + \alpha v^{n+1} + O(v^n + 2) & v \geq 0 \end{array} \right. .$$

Now consider a path

$$\begin{array}{ll} u & = at \\ v & = bt \end{array}$$

and consider the values of $f_i(u)g_j(v)$ for $t$ passing through zero. The difference in the $n$th derivative is a constant time $t^{n+1}$, and so the path is continuous in $n$th derivative, or $Cn$.

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*Bezier triangle theorems*

Prove that the functions

$$\frac{n!}{i!j!k!}u^i v^j w^k \,, \; i+j+k=n \,, \; i,j,k \geq 0$$

are non-negative over the triangle $u+v+w=1$, $u,v,w \geq 0$ and that they sum to unity.

• if $u,v,w \geq 0$, then powers of these variables are also non-negative and so must the product of such powers.

These functions are the expansions of the terms of $(u+v+w)^n$, and so their sum must be equal to $(1)^n = 1$.

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*Positions of maxima*

Identify the values of $u$ and $v$ ( in $[0..1]^2$ ) where

$$\frac{m!n!}{i!(m-i)!j!(n-j)!}(1-u)^{m-i}u^i(1-v)^{n-j}v^j$$

has its maximum.

Check this for $i = 0$ and $i = m$

• This function is the tensor product of univariate functions of $u$ and $v$. If we can find a point in the domain where these univariate functions have their independent maxima, that point will be the maximum of their product.

The maximum of a univariate function $f(u)$ over a limited domain is either at one end of that domain, or else at a point satisfying $df(u)/du = 0$

For $0 < i < m$, the derivative of $f = (1-u)^{m-i}u^i$ (the constant outside is always positive, so we can ignore it) is

$$\begin{aligned}
&-(m-i)(1-u)^{m-i-1}u^i + i(1-u)^{m-i}u^{i-1} \\
=&(1-u)^{m-i-1}u^{i-1}(-(m-i)u + i(1-u)) \\
=&(1-u)^{m-i-1}u^{i-1}(-mu + i)
\end{aligned}$$

which is zero at $u = 0$, $u = 1$ and $u = i/m$. The function itself is zero at $u = 0, 1$ and positive at $u = i/m$ and so this last is the maximum.

For $i = m$ (and the $i = 0$ case is exactly symmetrical) the function reduces to

$$f = u^m$$

which is zero at $u = 0$ and positive for all $u > 0$. The maximum is at $u = 1$, which happens to be equal to $u = i/m$ although $df/du$ is not zero there.

Thus the maximum of the tensor product function is at $u = i/m$, $v = j/n$.

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Identify the values of $(u, v, w)$ in $u + v + w = 1,\ u, v, w > 0$ where

$$\frac{n!}{i!j!k!}u^i v^j w^k$$

has its maximum.

The function $f = u^i v^j w^k$ must either have its maximum on the boundary of the domain, or else at a point where its derivatives with respect to $u$ and $v$, say (taking into account the dependence of the third variable on the two chosen) are both zero.

For $i, j, k > 0$, $f$ is zero on the boundary and positive in the interior, and we have

$$\begin{aligned}
df/du_v &= \partial f/\partial u - \partial f/\partial w \\
&= iu^{i-1}iv^j w^k - ku^i v^j w^{k-1} \\
&= u^{i-1}iv^j w^{k-1}(iw - ku)
\end{aligned}$$

which is zero in the interior at $u/i = w/k$. Similarly we derive $v/j = w/k$, and we may add these together to get

$$\frac{u}{i} = \frac{v}{j} = \frac{w}{k} = \frac{u+v+w}{i+j+k} = \frac{1}{n}$$

Hence $u = i/n$, $v = j/n$, $w = k/n$.

If $i = 0, j > 0, k > 0$ (and we may then deem the other boundaries covered by symmetry), the function reduces to $v^j w^k$, $j + k = n$ which is essentially the univariate case along the $u = 0$ boundary. By the previous example we know that the maxima along the edge are equally spaced. The derivative with respect to $u$ across the edge, with $v$ constant, is $\partial f/\partial u - \partial f/\partial w = -\partial f/\partial w = -kw^{k-1}$ which is negative for $k > 0, w > 0$, and so the maximum does indeed lie at the edge.

If $i, k = 0$, the function reduces to $f = v^n$ whose maximum lies at $u = 0$, $v = 1$, $w = 0$.

All of these case fit the formula $u = i/n$, $v = j/n$, $w = k/n$.

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For what values of $i, j, k,\ i + j + k = n$ do these maxima have their largest and smallest values for $n = 6$ ?

• The maxima are largest at the corners, where the function is unity, and smallest at the vertices nearest to $i = j = k$, which in the case of $n = 6$ is $i = j = k = 2$

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How large is the smallest maximum ?

• At the maximum $u = v = w = 2/6 = 1/3$ and so the function itself is equal to

$$f = \frac{6!}{2.2.2}(1/3)^6 = 10/81 \approx 0.12345678$$

**Parametric Surface Interrogations**

Design a parametric surface patch such that you can find points $Q$ so that
a) the nearest point is at a vertex
b) the nearest point is on an edge
c) the nearest point is in the interior
d) there is a point in the surface for which the Newton matrix is not positive definite

and report the position of an example $Q$ for each case.

Hints: You can keep things simple by letting $x = u$ and $y = v$ and just designing the variation of $z$. Assume the default boundaries $u = 0, 1$ $v = 0, 1$.

- The first three requirements place no constraints on the patch other than that it should not be singular or cyclic. We could just let $z = 0$, and then find points $Q$ near the vertices, edges and interior.

For the fourth constraint, there has to be some curvature, and we can achieve this easily with a nice symmetric bi-quadratic patch. Put the control points round the boundary at $z = 0$ and that in the middle at $z = 1$. The $u = 0.5$ isoparameter line then has the equation $z = 2x(1-x)$. The centre of curvature is at $x = .5, z = -.25$, and any point $Q$ on $x = y = .5$ with $z$ more negative than this has a maximum of distance at $x = y = .5, z = .25$.

Possible $Q$ points for the first three requirements then become, for example,
a) $x = y = -\epsilon, z = 0$
b) $x = -\epsilon,\ y = 0.5,\ z = 0$
c) $x = y = 0.5,\ z = 0.25 + \epsilon$

In fact there is a surface which is only bilinear which also has a point which could be a saddlepoint of distance. If we put the control points alternately at $z = 0$ and $z = 1$ around the square, the middle of the patch becomes a saddle for $Q$ points lying far enough above or below the surface. This has a Newton matrix with one eigenvalue positive and the other negative.

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