Subdivision Curve Definitions

Most of the course so far has been straightforward elaboration of a few key ideas.

•	coordinates	Descartes
•	parametric representations	Gauss
•	points not on a curve can	de Casteljau
	still be useful as design handles	Bezier
•	piecewise functions	Sommerfeld
		Schoenberg
		de Boor
•	curve as the limit	
	of a polygon	
•	divide and conquer	

Recursive subdivision curve theory is based on these last two.

In fact subdivision is the basis for interrogations: what we use for definitions is more properly called **refinement**.

The idea is that a polygon can be refined into one with more vertices, which represents exactly the same curve. As extra vertices are added, the polygon converges towards the curve. We can therefore make a piecewise linear approximation by this process. We do not have to evaluate points on the curve and join them together.

Examples

de Casteljau evaluation of points on a Bezier curve.

The first example of this was the order raising of a Bezier curve, whereby the polygon representing a curve of degree n can be used to construct a polygon representing the same curve as if it were of degree $n+1$. Each new vertex was a linear combination of two adjacent vertices of the old polygon.

This converges quite slowly. A more interesting construction is one which was originally devised to evaluate points on a Bezier curve.

Given: a polygon P^0 of $n+1$ vertices $P^0_{i\in 0..n}$

To Construct: the point at parameter $t \in 0 \ldots 1$

Construction:

Construct a new polygon P^1 of n vertices whose vertices are the weighted means in the ratio $(1-t):t$ of adjacent pairs of vertices of P^0

$$P^1_i = (1-t)P^0_i + tP^0_{i+1}\,,\ i \in 0\ldots n-1$$

Repeat this process to obtain a third polygon P^2 of $n-1$ vertices. Continue repeating until the result is a polygon of 1 vertex. This vertex is at the required point.

Proof: This is based on the fact that the basis functions are the terms in the expansion of $((1-t)+t)^n$. It can be made evident by laying out a tableau of intermediate results, as in the following example of a quadratic.

$$\begin{array}{ccccccccc}
P^0_0 & & & & P^0_1 & & & & P^0_2 \\
 & \backslash & & / & & \backslash & & / & \\
 & & P^1_0 & & & & P^1_1 & & \\
 & & & \backslash & & / & & & \\
 & & & & P^2_0 & & & &
\end{array}$$

A proof can be carried out by induction.

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What is interesting and slightly unexpected about this construction is that the flanks of the tableau - $[P^0_0,\ P^1_0,\ P^2_0]$ and $[P^2_0,\ P^1_1,\ P^0_2]$ are the polygons for two parts of the curve, so that the composite polygon $[P^0_0,\ P^1_0,\ P^2_0,\ P^1_1,\ P^0_2]$ is that for a **multiBezier** curve. Each piece can be similarly subdivided, and as the subdivision gets dense, the polygon tends towards the curve defined by P^0.

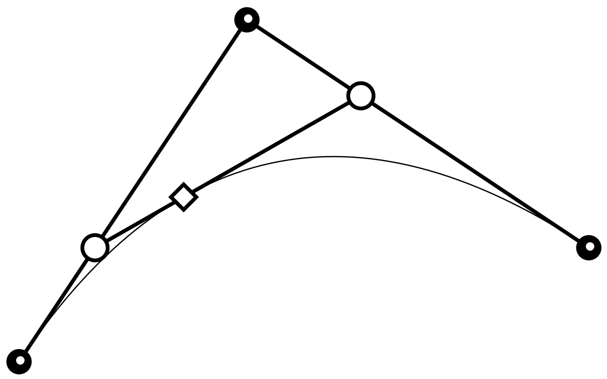

Figure 1. de Casteljau evaluation

de Boor recurrence for evaluation of points on an equal interval B-spline

The next example is also derived from an evaluation, this time of the B-spline.

Denote the B-spline basis function of degree n whose support ranges from j to $j+n+1$ by $f^n_j(t)$.

Lemma: The magnitudes of the discontinuities of nth derivative in the degree n equal interval B-spline basis function are distributed as the terms of the expansion of $(1-1)^{n+1}$.

i.e. the discontinuity of the nth derivative of $f_0^n(t)$ at $t=i$ is given by $(-1)^i\left(\frac{(n+1)!}{i!(n+1-i)!}\right)$.

The proof of this is by induction. It is true for $n=0$.

Suppose that it is true for n. We show that it is also true for $n+1$.

We know from the hodograph property that the derivative of $f_0^{n+1}(t)$ is $f_0^n(t)-f_1^n(t)$.

Therefore the discontinuities of $f_0^{n+1}(t)$ are given by

$$
\begin{aligned}
&(-1)^i\left(\frac{(n+1)!}{i!(n+1-i)!}\right)-(-1)^{i-1}\left(\frac{(n+1)!}{(i-1)!(n+1-(i-1))!}\right)\\
=&(-1)^i\left(\frac{(n+1)!(n+2-i)}{i!(n+2-i)!}\right)-(-1)^{i-1}\left(\frac{(n+1)!i}{(i)!(n+2-i))!}\right)\\
=&\frac{(-1)^i(n+1)!}{i!(n+2-i)!}((n+2-i)+i)\\
=&\frac{(-1)^i(n+2)!}{i!(n+2-i)!}
\end{aligned}
$$

■

Theorem de Boor recurrence for B-spline evaluation

Given: the function $\phi(t)=tf_0^n(t)+(n+2-t)f_1^n(t)$

To Prove: that $\phi(t)=f_0^{n+1}(t)$

Proof: $\phi(t)$ is a piecewise polynomial of degree $n+1$, whose support is $n+2$ spans. It would appear to have discontinuities of nth derivative from this definition.

However, the discontinuity of nth derivative at $t=i,\ i\in 0\ldots n+2$ is given by

$$
\begin{aligned}
&i(-1)^i\frac{(n+1)!}{i!(n+1-i)!}+(n+2-i)(-1)^{(i-1)}\frac{(n+1)!}{(i-1)!(n+2-i)!}\\
=&(-1)^i\left(\frac{(n+1)!}{(i-1)!(n+1-i)!}-\frac{(n+1)!}{(i-1)!(n+1-i)!}\right)\\
=&0
\end{aligned}
$$

Thus it does not have discontinuities of nth derivative. Unless identically zero, it must therefore be $f_0^{n+1}(t)$

However, by induction we can show that $f_j^k(t)>0$ over the interior of its support, and so $\phi(t)$ is also greater than 0.

■

This means that we can evaluate $f_0^{n+1}(t)$ by first evaluating $f_0^n(t)$ and $f_1^n(t)$ which equals $f_0^n(t-1)$.

Because of the linearity of the scheme, we can use this result in two ways:

- we can evaluate basis functions at a given value of t and then use them in the definition of a B-spline curve as a sum of coefficients times basis functions.

- we can set up a tableau very similar to that found in the de Casteljau construction, but with slightly different factors in the linear combinations, for direct operation on the polygon. Exactly as in the de Casteljau construction, this gives a new polygon which is a refined version of the old one. The increase in the number of vertices is equal to the number of knots inserted.

Knot insertion in an equal interval B-spline

Evaluation by the de Boor recurrence is equivalent to adding knots until the multiplicity is sufficient to give a 'Bezier end condition'. One stage of the recurrence is equivalent to adding one knot at t.

For regular refinement of the whole of a curve it is convenient to add single knots at the middles of all the intervals, and we can derive the mask very straightforwardly in a single process.

The nth derivative of a B-spline is piecewise constant, and we can determine the values in each of the spans by taking the hodograph n times. i.e. by taking the nth differences of the control points.

When we insert knots at the mid points, the geometry is unchanged, and so the nth derivatives are the same on both sides of the new knots. We can therefore write down the desired differences by just repeating the values twice each. (we also have to divide by 2^n to adjust for the abscissa scale).

Finally, the necessary polygon to give the required nth differences is obtained by anti-differencing.

As an example we determine the effect of knot halving on a quadratic B-spline basis function.

The basis function is exactly that whose polygon has a single unit value, all other control points having the value zero.

```
original values      ...0   0   1   0   0...
1st differences           0   1   -1  0
2nd differences             1   -2  1

repeated values        1   1   -2  -2  1   1
1st antidiff.            1   2   0   -2  -1
2nd antidiff.              1   3   3   1
```

These values need to be divided by $4 = 2^2$

Generic Refinement

The above have given examples of ways in which a new polygon can be created from an old, in such a way that increasing the number of vertices in the polygon leads to it converging towards the curve which the original polygon represents.

In general we can set up a linear combination matrix M by which we multiply the original polygon P_0 (thought of as a column vector of vertices) to give a new polygon P_1.

$$P_1 = MP_0$$

For the regular knot insertion process we find that the columns are all the same, but shifted by the number of spans which each old span is divided into. This number is called the **arity** of the subdivision. The total number of entries in a column between the top non-zero entry and the bottom one (inclusive) is called the **width**. The column itself is called the **mask**, and the entries in it are called the **coefficients**.

The rows are called **stencils** and there are as many different ones as the arity.

The arity and the mask (in which the width is implicit) together define the refinement process completely.

For the particular quadratic B-spline process described above, the matrix M takes the form

$$M = \frac{1}{4}\begin{bmatrix} \ddots & & & & & \\ & 3 & 1 & & & \\ & 1 & 3 & & & \\ & 0 & 3 & 1 & & \\ & 0 & 1 & 3 & & \\ & 0 & 0 & 3 & 1 & \\ & 0 & 0 & 1 & 3 & \\ & & & & & \ddots \end{bmatrix}$$

However, we can define other processes, which are not B-spline knot insertion, but which still give column-circulant refinement matrices. Indeed, we can ask what properties the mask must have to give a well-behaved limit curve if applied infinitely often to an initial polygon, and how the properties of the limit curve relate to those of the mask.

This freedom opens up a wide range of possible curves. The equivalent expansion of horizons in the surface case is even more dramatic, but it is interesting enough for curves.

In the continuation of these notes we shall explore two specific families of subdivisions, the Box-splines (equal to the B-splines in the univariate case, but more general in the bivariate case, and giving a delightful geometric insight into the properties of these curves) and a family of interpolating schemes.

In the notes on analysis of subdivision schemes we shall explore the ways in which the properties of the mask and those of the limit curve can be related, thus allowing the design of masks to give desired limit curve properties.