Subdivision Curve Definitions

These notes are a continuation of those on the nature of subdivision curves previously issued. We start with some small print defining things, then classify curve subdivisions according to the pattern the refinement makes, and then explore two major families of refinements. These two families do not exhaust the possiblilities, but provide a base from which to work.

Limitations to the treatment here

We limit ourselves here to:-

- stationary schemes, in which the same mask is used in all iterations.

  The same mask is also used at all places of the curve. In my view the eventual standard methods will not be stationary in either sense, the mask being dependent on the geometry of the defining control polygon. However, such schemes will tend towards a stationary limit as the recursion proceeds, and so understanding the stationary case is an essential grounding.

- symmetric schemes, in which the mask is palindromic.

  Making the mask depend on the local geometry will also spoil this property, but we can define the dependence in such a way that this property, also, applies in the limit.

Characterisation of subdivision schemes

The scheme is basically defined by the matrix equation

$$P' = S.P$$

where S is higher than it is wide, so that P' contains more points than P.

S is called the **subdivision matrix**.

All the columns of S are identical (except as perturbed by end conditions), but shifted by a number of rows called the **arity**. The arity gives the number of new polygon edges corresponding to each old edge.

Each column contains a sequence of numbers called the **coefficients** and the number of them is called the **width** of the scheme.

The column is called the **mask**, or the **coefficient map**.

The rows of S are called the **stencils**, or **templates**. There are arity different rows, and so it is clearest to regard the matrix S as being defined by the mask, (its columns) and the arity.

The stencils are then implicit.

Most schemes have coefficients which are ratios of small numbers, and it is convenient to take the least common multiple of their denominators outside the matrix, which can then be recorded in print as a matrix of integers with a common denominator outside. This makes the printed form much less fussy.

Four Patterns

For univariate schemes there are four patterns, depending on whether the width, w, and the arity, a, are even or odd.

w odd, a even

If the width is odd and the arity even, we find that each old vertex finds itself opposite a new vertex. Each old mid-edge is also opposite a new vertex.

We call such schemes **primal**.

w even, a even

If the width is even and the arity even, we find that each old vertex finds itself opposite a new mid-edge. Each old mid-edge is also opposite a new mid-edge.

We call such schemes **dual**.

w odd, a odd

If the width is odd and the arity odd, we find that each old vertex finds itself opposite a new vertex. Each old mid-edge is opposite a new mid-edge.

We call such schemes **both**, since both vertices and mid-edges preserve their nature under subdivision.

w even, a odd

If the width is even and the arity odd, we find that each old vertex finds itself opposite a new mid-edge, and each old mid-edge is opposite a new vertex.

We call such schemes **neither**.

Squaring

If a scheme is applied twice we may regard the effect as being that of a new scheme, called the **square** of the original.

Exercise

Determine the width and arity of the square, as functions of the width and arity of the original scheme. Hence determine the classification of the square.

Box-Splines

Box-Splines are a generalisation of equal interval B-splines to multiple dimensions, which coincide with the equal interval B-splines in the case of univariate functions.

They were observed by Sommerfeld, when exploring the nature of the distribution of the sum of large numbers of random variables, and he used them to prove that this distribution was the Gaussian.

A **Box** is the outer product of d intervals, and we can imagine it as a hypercube in a space of d dimensions.

Consider a subspace of $d-1$ dimensions, perpendicular to the main diagonal of the box, and take the hypervolume of its intersection with the box. Consider this volume as a function of the position in which the subspace cuts the diagonal. As the subspace moves, the size and the shape of the intersection alter, but it turns out that the volume is a C^{d-2} piecewise polynomial of degree $d-1$, non-zero only inside a region of d pieces.

The functions by which the volume varies are therefore exactly the equal interval B-splines.

This visualisation of the B-splines leads directly to two standard results:

- Because we may visualise a row of boxes forming a beam of constant cross-sectional 'area', we see at once that the sum of the translates of the B-spline is constant. By suitable choice of area units we can make it unity.
- All 'areas' are non-negative.

Refinement of Box-splines

We can also see that Box-splines must support a refinement definition, because an interval can always be subdivided into two equal halves. The outer product of such divisions cuts the Box into 2^d sub-boxes, each similar to the original. If we are refining by a factor of a rather than by 2, the box is split into a^d sub-boxes.

This makes evident that a box-spline function can be expressed as the sum of appropriate weights times the narrower copies of the functions associated with the sub-boxes.

$$f_i(t) = \sum_j S_{ij} f'_j(t)$$

Now when we use these functions to define a curve we do so by an equation of the form

$$\begin{aligned} P(t) &= \sum_i P_i f_i(t) \\ &= \sum_i \sum_j P_i S_{ij} f'_j(t) \\ &= \sum_j \sum_i P_i S_{ij} f'_j(t) \\ &= \sum_j P'_j f'_j(t) \end{aligned}$$

provided that $P'_j = \sum_i P_i S_{ij}$.

The equation linking the basis functions before and after subdivision gives the equation linking the control points before and after.

Because of the translational invariance of the functions (due to our equal interval limitation), all the columns of S are the same, but shifted through an appropriate shift, the arity.

Determining S

The numbers in a column can be determined by counting the number of subcells in each cross section of the subdivided box.

It is easier, however, to determine them by use of a recurrence. Suppose that we have the mask for dimension d. Then we may form a box of dimension $d+1$ by assembling a extrusions of the subdivided box of dimension d.

The first row has exactly the counts of the d-mask; the second the same mask shifted by 1; the third shifted by 2, etc., and the $d+1$-mask can be computed by just summing these contributions.

There is an operation which has exactly the same pattern: this is the long multiplication of polynomials. We can invoke this similarity by defining the **generating function** of a sequence of values s_i to be the polynomial $\sum_i s_i z^i$.

The generating functions of the binary box-splines are therefore

$$(1+z)^d/2^d,$$

and those of a-ary ones

$$[(1-z)^a/(1-z)]^d/a^d$$

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The Dubuc-Deslauriers interpolating refinements

For some purposes it is more convenient to be able to define a curve which passes through some points that we know.

This can be achieved in a refinement context by arranging that 1 in a of the stencils (rows of the subdivision matrix) should contain a single unit value and all the rest zero.

The intermediate rows can be chosen to give good properties for the refinement.

Dubuc and Deslauriers chose to use Lagrange interpolation for their schemes.

If we use 2-point Lagrange, the scheme results in the polygon joining the original points.

If we use 4-point Lagrange, the binary scheme results in a C1 curve through the points.

If we use 6-point Lagrange, the binary scheme results in a C2 curve through the points.

But the pattern stops here. Higher Lagrange schemes do increase the continuity but only very slowly.

However, Lagrange is not the only option open to us. The 4-point scheme is actually very nearly C2 (in a sense which will be explained later in the course), and a 4-point ternary scheme can be set up which is C2 for a rather narrow range of coefficients. The Lagrange solution does not lie in this range.

Interpolating schemes for other patterns

The Dubuc-Deslauriers binary family gives only primal subdivisions.

The same idea, of imposing the interpolation requirement by forcing one of the stencils to contain a unit element, can be applied in the 'primal' and 'both' scenarios. Choice of the other stencils is then made by identifying equations between the coefficients which give desirable properties.

In the 'neither' case, the square is 'both', and so the conditions can be set up for the square and then echoed back into the single stage. This involves a set of quadratic equations, but leads to a ternary 4-point scheme which is only C1 and in fact behaves rather badly.

In the 'dual' case, however, the curve never has a new vertex corresponding to the original vertices at any stage, and so we cannot make an interpolating scheme quite so easily.

I think I have a proof that it is not possible to make a binary dual interpolating scheme, but this is not properly articulated. Whether dual interpolating schemes of higher arity exist is an open question.