Subdivision Surfaces: exercises

Biquadratic subdivision

Apply two steps of biquadratic subdivision to a cube.

You will need some coefficients for the triangle face. Use the stencil

$$\begin{array}{cc} 1 & \\ 4 & 1 \end{array} /6$$

and its rotations.

- Use a cube of side 24 to make the sums easier, centred on the origin.

The initial step gives new vertices in each face, of coordinates

$$(\pm 12, \pm 12, \pm 24)$$

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There are six square faces corresponding to the original faces, twelve rectangular faces corresponding to the original edges, and eight three-sided faces corresponding to the original vertices.

At the second step each of the face-faces gives a face with corners like $(24, \pm 6, \pm 6)$; each of the edge-faces gives a face with corners like $(21, 15, \pm 6), (15, 21, \pm 6)$; each of the triangular vertex-faces gives a new triangular face with corners like $(14, 14, 20)$.

There are 48 additional rectangular faces and 24 quad faces completing the figure.

Thus the number of triangular faces remains at 8. while the number of quad faces has risen to 90.

Apply two steps of biquadratic subdivision to a coarse toroidal shape with 16 vertices.

- Assume that the vertices of the original polyhedron are at

$(\pm 8, \pm 8, \pm 8)$ and $(\pm 24, \pm 24, \pm 8)$

There are square faces in $x = \pm 8$ and $y = \pm$ and four tapered (keystone) quad faces in each of $z = 8$ and $z = -8$.

After one step the square faces contain new faces like $(8, \pm 4, \pm 4)$ and the keystone faces new faces like $(\pm 6, 12, 8), (\pm 10, 20, 8)$.

New quad faces join these, making 64 in all. They effectively form 8 rings of eight, in each of the directions of the torus.

After the second step there are 256 quad faces. No triangles have appeared.

Box-splines

Expand out the generating function

$$(1+x)^2(1+y)^2(1+xy)^2$$

to give a mask. What are the stencils ?

-

$$(1+x)^2(1+y)^2 = \begin{array}{ccc} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{array}$$

$$(1+x)^2(1+y)^2(1+y)^2 = \begin{array}{ccccc} & & 1 & 2 & 1 \\ & 2 & 4 & 4 & 2 \\ 1 & 4 & 6 & 4 & 1 \\ 2 & 4 & 4 & 2 & \\ 1 & 2 & 1 & & \end{array}$$

There are three directions of edges, each with a new vertex logically in the middle. These have the stencils

$$\begin{array}{ccc} & & 2 \\ 4 & * & 4 \\ 2 & & \end{array}, \quad \begin{array}{ccc} & 4 & 2 \\ & * & \\ 2 & 4 & \end{array} \quad \text{and} \quad \begin{array}{ccc} 2 & & 4 \\ & * & \\ 4 & & 2 \end{array}.$$

The stencil for the new vertex-vertex is

$$\begin{array}{ccc} & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & \end{array}$$

Note that all of the stencils sum to 12.

Expand out the generating function

$$(1+x+x^2)^2(1+y+y^2)^2(1+xy+x^2y^2)^2$$

to give a mask. What are the stencils ?

- $$(1+x+x^2)^2 = 1 + 2x + 3x^2 + 2x^3 + x^4 = [\,1 \quad 2 \quad 3 \quad 2 \quad 1\,]$$

$$(1+x+x^2)^2(1+y+y^2)^2 = \begin{array}{ccccc} 1 & 2 & 3 & 2 & 1 \\ 2 & 4 & 6 & 4 & 2 \\ 3 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 4 & 2 \\ 1 & 2 & 3 & 2 & 1 \end{array}$$

$$(1+x+x^2)^2(1+y+y^2)^2(1+xy+x^2y^2)^2 = \begin{array}{ccccccccc} & & & & 1 & 2 & 3 & 2 & 1 \\ & & & 2 & 6 & 10 & 10 & 6 & 2 \\ & & 3 & 10 & 20 & 24 & 20 & 10 & 3 \\ & 2 & 10 & 24 & 36 & 36 & 24 & 10 & 2 \\ 1 & 6 & 20 & 36 & 45 & 36 & 20 & 6 & 1 \\ 2 & 10 & 24 & 36 & 36 & 24 & 10 & 2 & \\ 3 & 10 & 20 & 24 & 20 & 10 & 3 & & \\ 2 & 6 & 10 & 10 & 6 & 2 & & & \\ 1 & 2 & 3 & 2 & 1 & & & & \end{array}$$

Determining the stencils requires a little care. This is a ternary scheme, and so the terms corresponding to the new vertices must be those with values 6 and 45.

The three logical positions for new vertices are:-

- at a vertex

- one third of the way along an edge

- at the centre of a face.

The three stencils are

$$\begin{array}{cccc} & & 6 & 6 \\ \text{vertex -} & 6 & 45 & 6 \\ & 6 & 6 & \end{array}$$

$$\begin{array}{ccccc} & & 2 & & 10 \\ \text{edge -} & 1 & 36 & * & 20 \\ & 2 & 10 & & \end{array}$$

appearing in 6 different orientations

$$
\begin{array}{ccccc}
 & 3 & & 24 & 3 \\
 & & * & & \\
\text{face -} & 24 & & 24 & \\
 & 3 & & &
\end{array}
$$

appearing in two orientations.

Note that all of these stencils sum to 81

Others

Determine the mask which results when

$$
\begin{array}{cccccccc}
 & & -1 & & -1 & & & \\
 & 0 & & 0 & & 0 & & \\
-1 & & 10 & & 10 & & -1 & \\
 & 0 & & 32 & & 0 & & /32 \\
-1 & & 10 & & 10 & & -1 & \\
 & 0 & & 0 & & 0 & & \\
 & & -1 & & -1 & & &
\end{array}
$$

is applied twice.

Remember that this is a $\sqrt{2}$ scheme, and so the configuration needs to be rotated through 45 degrees between the two steps.

- I think the answer is

$$
\begin{array}{ccccccccccc}
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & -9 & & -9 & & 1 & & \\
 & 1 & & -19 & & -40 & & -19 & & 1 & \\
1 & & -19 & & 50 & & 50 & & -19 & & 1 \\
 & -9 & & 50 & 320 & 502 & 320 & 50 & & -9 & \\
2 & & -40 & & 502 & 1024 & 502 & & -40 & & 2 \\
 & -9 & & 50 & 320 & 502 & 320 & 50 & & -9 & \\
1 & & -19 & & 50 & & 50 & & -19 & & 1 \\
 & 1 & & -19 & & -40 & & -19 & & 1 & \\
 & & 1 & & -9 & & -9 & & 1 & & \\
 & & & 1 & & 2 & & 1 & & &
\end{array}
$$

Each of the stencils sums to 1024.

Determine the mask which results when

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array} /8$$

is applied twice.

Remember that this is a $\sqrt{2}$ scheme, and so the configuration needs to be rotated through 45 degrees between the two steps.

- $$\begin{array}{ccccccc} & & 1 & 2 & 1 & & \\ & 2 & 6 & 8 & 6 & 2 & \\ 1 & 6 & 14 & 18 & 14 & 6 & 1 \\ 2 & 8 & 18 & 24 & 18 & 8 & 2 \\ 1 & 6 & 14 & 18 & 14 & 6 & 1 \\ & 2 & 6 & 8 & 6 & 2 & \\ & & 1 & 2 & 1 & & \end{array}$$

Generation of all subdivision schemes

Sketch one example of each of the eight categories possible within the combinations of quad, tri and hex grids and primal, dual, both and neither types.

Draw the new grid in one colour and the old one in another.

Sketch three examples of schemes with complex arity, one on a quad grid, one on a tri-grid and one on a hex-grid.