**Part III - Lent Term 2004**

# Approximation Theory – Lecture 2

## 2 Weierstrass theorems

The following two theorems lie at the heart of approximation theory.

**Theorem 2.1 (Weierstrass[1][1885])** *For any finite $I = [a, b]$, the set $\mathcal{P}$ of all algebraic polynomials is dense in $C(I)$, i.e., for each $f \in C(I)$ and for each $\varepsilon$ there exists some $p \in \mathcal{P}$ with*

$$|f(x) - p(x)| < \varepsilon, \quad a \le x \le b.$$

**Theorem 2.2 (Weierstrass [1885])** *The set $\mathcal{T}$ of all trigonometric polynomial is dense in $C(\mathbb{T})$.*

**Comment 2.3** Weierstrass brought two news to the mathematical world (as usual: a bad one and a good one). The first from 1872 shocked the mathematical community: *there exist functions in $C[a, b]$ which are not differentiable at every point of $[a, b]$*. The second result appeared in 1885 and stated above is in a sense the converse. Thus the set of continuous functions contains very, very non-smooth functions, but they can each be approximated arbitrarily well by the ultimate in smooth fuctions. (Extracts are taken from a recent and very nice survey by A Pinkus, J. Approx. Theory 107 (2000), 1-66.)

Weierstrass theorems (and in fact their original proofs) postulate existence of *some* sequence of polynomials converging to a prescribed continuous function uniformly on a bounded closed intervals. The proofs below provide an explicit construction for each case.

### 2.1 Korovkin theorem on positive linear operators

**Definition 2.4 (Positive operators in $C(K)$)** Let $C(K)$ be the set of *real-valued* continuous maps on a compact $K$. For those, there is a natural (partial) order: $f \ge g$ means $f(x) \ge g(x)$ for all $x \in K$. An operator $U : C(K) \to C(K)$ is called *positive* if $f \ge 0$ implies $U(f) \ge 0$ and it is called *monotone* if $f \ge g$ implies $U(f) \ge U(g)$. If $U$ is linear then it is positive iff it is monotone.

**Example 2.5** Important examples of linear positive operators on $C[a, b]$ are given by

$$U_n(f, x) = \textstyle\int_a^b K_n(x, t) f(t)\, dt, \quad \text{with a positive kernel } K_n(x, t) \ge 0,$$

or its discrete analogue $U_n(f, x) = \sum_{i=1}^n k_{n,i}(x) f(t_i)$.

**Theorem 2.6 (Korovkin[2] [1957])** *Let $K$ compact, $(U_n)$ in $\mathcal{L}(C(K))$ and positive. Assume that there exist finite sets $(a_i), (p_i) \in C(K)$ such that*

$$p(x, t) := p_t(x) := \sum_{i=1}^m a_i(t) p_i(x) \ge 0 \quad \textit{with equality iff } x = t.$$

*If $U_n(p_i) \to p_i$ on the set $F := (p_i)$, then $U_n(f) \to f$ for any $f \in C(K)$.*

**Example 2.7** Two classical examples are

$$\begin{aligned} K &= [a, b], & F &= (1, x, x^2), & p_t(x) &= (x - t)^2; \\ K &= [0, 2\pi, ) & F &= (1, \cos x, \sin x), & p_t(x) &= 1 - \cos(x - t). \end{aligned}$$

[1] Karl Weierstrass, 1815-1897, he is known, e.g., by Bolzano-Weierstrass theorem, the M-test for convergence, but he is also the inventor of epsilontics: "for any $\varepsilon > 0$ there exists a $\delta > 0 \ldots$"

[2] Pavel Korovkin, 1913-1985, he had a turbulent start of his scientiic career: PhD in 1939, in 1941-45 in the combat service (artillery), but already in 1947 he has got the senior doctor degree (Habilitation).

**Proof.** The idea of the proof is that, given $f \in C(K)$ and $\varepsilon > 0$, we can construct for any $t \in K$ two polynomials $q_t^+, q_t^- \in \operatorname{span}(F)$ s.t.

$$1) \quad q_t^- < f < q_t^+ \quad \text{on} \quad K, \qquad 2) \quad |q_t^+(x) - q_t^-(x)|_{x=t} < \varepsilon, \qquad 3) \quad U_n(q_t^\pm) \to q_t^\pm \quad \text{uniformly in } t.$$

The monotonicity of $U_n$ provides

$$1') \quad U_n(q_t^-) < U_n(f) < U_n(q_t^+),$$

while the convergence (3) (coupled with (2) in $(2')$ below) ensures that, for sufficiently large $n$, *independently of* $t$,

$$2') \quad |U_n(q_t^+, t) - U_n(q_t^-, t)| < \varepsilon', \qquad 3') \quad |U_n(q_t^\pm, t) - q_t^\pm(t)| < \varepsilon''.$$

Hence, for any $t \in K$,

$$|U_n(f,t) - f(t)| \le |U_n(f,t) - U_n(q_t^-, t)| + |U_n(q_t^-, t) - q^-(t)| + |q_t^-(t) - f(t)| \le \varepsilon' + \varepsilon'' + \varepsilon\,.$$

**Particular case.** Construction of such $q_t^\pm$ in general situation is given in §2.3 as a (non-examinable) supplement (for those interested), but here we consider only one important particular case

$$K = [a,b], \quad F = (1, x, x^2), \quad p_t(x) = (x-t)^2.$$

Take any $\varepsilon > 0$. Then, because $K$ is a compact, any $f$ continuous on $K$ is uniformly continuous, i.e., there is a $\delta$ (which depends on $f$) such that

$$|x - t| < \delta \quad \Rightarrow \quad |f(x) - f(t)| < \varepsilon\,. \tag{2.1}$$

For any $t \in K$, we define the polynomials $q_t^\pm$ (in $x$) as follows

$$q_t^\pm(x) := f(t) \pm \left(\varepsilon + 2\|f\| \frac{(x-t)^2}{\delta^2}\right).$$

Let us verify that these $q_t^\pm$ satisfy conditions (1)-(3) above.

1) We have

$$|x - t| < \delta \quad \Rightarrow \quad f(x) - q_t^+(x) \le |f(x) - f(t)| - \varepsilon \overset{(2.1)}{<} 0\,,$$

while

$$|x - t| \ge \delta \quad \Rightarrow \quad f(x) - q_t^+(x) < |f(x) - f(t)| - 2\|f\| \le 0\,.$$

2) Clearly,

$$|q_t^+(x) - q_t^-(x)|_{x=t} = 2\varepsilon.$$

3) We can represent both polynomials in the form

$$q_t(x) = c_2(t)p_2(x) + c_1(t)p_1(x) + c_0(t)p_0(x)\,, \quad p_i(x) = x^i,$$

where $c_i(t)$ are uniformly bounded functions, say, $|c_i(t)| \le c_\varepsilon(f)$, hence

$$\|U_n(q_t^\pm) - q_t^\pm\| \le 3c_\varepsilon(f) \max_i \|U_n(p_i) - p_i\|,$$

and because of convergence $U_n(p_i) \to p_i$ we can find an $n_0$ (that depends on $\varepsilon$ and $f$) such that

$$\|U_n(p_i) - p_i\| \le \varepsilon / c_\varepsilon(f), \quad n \ge n_0\,,$$

whence $\|U_n(q_t^\pm) - q_t^\pm\| \le 3\varepsilon$. ∎

**Corollary 2.8** *Let $(U_n)$ be in $\mathcal{L}(C[a,b])$ and positive. Then*

$$U_n(p_i) \to p_i \quad \text{on} \quad F = \{1, x, x^2\} \quad \Rightarrow \quad U_n(f) \to f \quad \forall f \in C[a,b].$$

**Corollary 2.9** *Let $(U_n)$ be in $\mathcal{L}(C(\mathbb{T}))$ and positive. Then*

$$U_n(p_i) \to p_i \quad \text{on} \quad F = \{1, \cos x, \sin x\} \quad \Rightarrow \quad U_n(f) \to f \quad \forall f \in C(\mathbb{T}).$$

## 2.2 Exercises

**2.1.** Using Weierstrass theorem prove that the polynomials are dense in $C^k[0,1]$, the Banach space of all $k$ times continuously differentiable functions on $[0,1]$ with the norm

$$\|f\|_\infty^{(k)} := \max_{0\le i\le k} \|f^{(i)}\|_\infty \,.$$

**2.2.** Prove that, for any positive linear operator $U$, we have $\|U\| = \|U(1,\cdot)\|$. Then derive that, under assumption of Korovkin theorem, we have

$$\sup_n \|U_n\| < M < \infty.$$

(The latter inequality is in fact a *necessary* condition for any sequence $(U_n)$ of linear operators to provide convergence $U_n(f) \to f$ for *all* $f$ in $C[a,b]$.)

*Hint.* Apply $U$ to the functions in the inequality $\quad -\|f\| \le f \le \|f\|$.

**2.3.** (*Exam question 2002*) Use Korovkin theorem for the case $K = [a,b]$ and $p(x,t) = (x-t)^2$ to show that the only linear positive operator $U : C[a,b] \to C[a,b]$ such that

$$U(p_i) = p_i \quad \text{on} \quad F = \{1, x, x^2\}$$

is the identity operator, i.e. $U(f) = f$ for all $f \in C[a,b]$.

## 2.3 General construction of $q_t^\pm$

We generalize the construction used for $p_t(x) = (x-t)^2$, it is useful to compare the corresponding steps.

1) From the assumption, $\mathrm{span}(F)$ contains *strictly* positive polynomials, e.g., $p_{t'} + p_{t''}$ for any fixed $t' \neq t''$. Let $p^*$ be one such. For any $t \in K$, set

$$f =: \frac{f(t)}{p^*(t)} p^* + h_t.$$

This equality is simply the formula of interpolation of $f$ by $p_*$ at one point $x = t$ with the remainder $h_t$. It defines a continuous function $h$ of two variables such that

$$h(x,t) := h_t(x) \in C(K^2), \quad h(t,t) = 0, \quad \forall t \in K.$$

Take any $\varepsilon > 0$. Then

$$|h| \leq \varepsilon + \text{a bound for } |h| \text{ on the set } \Delta := \{(x,t) : \ |h| \geq \varepsilon\}.$$

Since $h$ is continuous, this set is closed, hence compact, it also does not contain the set $\{(t,t) : \ t \in K\}$, the only zero-set of $p$. Therefore, with $\delta := \min_\Delta p$, we have $\delta > 0$ and $|h| \leq \|h\| \leq \frac{\|h\|}{\delta} p =: \gamma p$ on $\Delta$. So

$$|h| \leq \varepsilon + \gamma p, \quad \text{hence} \quad |h_t| \leq \varepsilon + \gamma p_t \quad \text{uniformly in } t \in K.$$

2) It is almost what we need with the exception that $\varepsilon$ (i.e., the constants) may not belong to $\mathrm{span}(F)$. But we can majorize $\varepsilon$ by the positive polynomial $\varepsilon\alpha p_*$ with $\alpha := 1/\min_x p_*(x) < \infty$. Thus, the polynomials

$$q_t^\pm := \frac{f(t)}{p_*(t)} p_* \pm \left(\varepsilon\alpha p_* + \gamma p_t\right)$$

satisfies the inequalities $q_t^- < f < q_t^+$ and (since $p_t(t) = 0$)

$$|q_t^+(t) - q_t^-(t)| = 2\varepsilon\alpha p_*(t) < 2\varepsilon\alpha \|p_*\| =: \varepsilon_1$$

3) By assumption, $U_n \to 1$ on $F$, hence also on $\mathrm{span}(F) := \{\sum_{i=1}^m c_i f_i \ : \ c \in \mathbb{R}^m\}$. The latter is finite-dimensional, therefore $U_n \to 1$ uniformly on bounded subsets of $\mathrm{span}(F)$. The subset $P = \{p_t\}_{t \in K}$ is bounded, thus $U_n \to 1$ on $P$, i.e., $U_n(p_t) \to p_t$ uniformly in $t$, in particular $U_n(p^*) \to p^*$, hence

$$U_n(q_t^\pm) \to q_t^\pm \quad \text{uniformly in } t.$$

■