Part III - Lent Term 2005
Approximation Theory – Lecture 3

3 Weierstrass theorems (cont.)

3.1 Bernstein polynomials and the first Weierstrass theorem

Definition 3.1 (Bernstein[1] polynomials [1912]) For $f \in C[0,1]$, the Bernstein polynomials of f are given by the formula

$$B_n(f,x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(\tfrac{k}{n}). \tag{3.1}$$

This formula produces a positive linear map $f \to B_n(f)$ of $C[0,1]$ into $\mathcal{P}_n$.

Lemma 3.2 *For any $n, m \in \mathbb{N}$, we have $B_n(\mathcal{P}_m) \subseteq \mathcal{P}_m$.*

Proof. We prove the statement once we show that $D^m B_n(\mathcal{P}_m) \equiv 0$. So, let us find the derivative of $B_n(f)$:

$$\begin{aligned} B_n'(f,x) &= \textstyle\sum_{k=1}^{n} \binom{n}{k} k x^{k-1}(1-x)^{n-k} f(\frac{k}{n}) - \sum_{k=0}^{n-1} \binom{n}{k}(n-k) x^k (1-x)^{n-k-1} f(\frac{k}{n}) \\ &= n \textstyle\sum_{k=0}^{n-1} \binom{n-1}{k} x^k (1-x)^{n-1-k} \Delta_{1/n} f(\frac{k}{n}), \end{aligned} \tag{3.2}$$

with $\Delta_{1/n} f(t) := f(t+\frac{1}{n}) - f(t)$. Since $\Delta_h(\mathcal{P}_m) \subseteq \mathcal{P}_{m-1}$, one obtains that $\Delta_h^{m+1} := \Delta_h \Delta_h^m$ vanishes identically on $\mathcal{P}_m$, hence $B_n(\mathcal{P}_m) \subseteq \mathcal{P}_m$. □

Lemma 3.3 *$B_n(p_i) \to p_i$ for $p_i(x) = x^i$, where $i = 0, 1, 2$.*

Proof. Since $B_n(\mathcal{P}_m) \subseteq \mathcal{P}_m$, and since also $B_n(f,x) = f(x)$ at $x = 0, 1$ for any f, it follows that $B_n(p) = p$ for $p \in \mathcal{P}_1$, thus $B_n(x^i) \to x^i$ for $i = 0, 1$ trivially and we are done once we show that $B_n(q) \to q$ for some $q \in \mathcal{P}_2 \setminus \mathcal{P}_1$, e.g., for $q(x) = x(1-x)$.

This q vanishes at 0 and 1 and is quadratic, hence so is $B_n(q)$, therefore $B_n(q) = \gamma q$. It follows that $B_n'(q,0) = \gamma q'(0) = \gamma$, while by (3.2), $B_n'(q,0) = n(q(\frac{1}{n}) - q(0)) = nq(\frac{1}{n}) = 1 - \frac{1}{n}$, i.e., $\gamma = 1 - \frac{1}{n}$. Thus $B_n(q) = (1 - \frac{1}{n})q \to q$. □

Theorem 3.4 (Weierstrass [1885]) *For any finite $I = [a,b]$, $\mathcal{P}$ is dense in $C(I)$, i.e., for each $f \in C(I)$ and for each $\varepsilon > 0$ there exists some $p \in \mathcal{P}$ such that*

$$|f(x) - p(x)| < \varepsilon, \quad a \le x \le b.$$

Proof. For $I = [0,1]$, the Korovkin Theorem, with the choices

$$F = \{1, x, x^2\} \in \mathcal{P}_2, \quad p(x,t) = (x-t)^2, \quad U_n = B_n.$$

shows that $B_n(f) \to f$ for all $f \in C[0,1]$. □

Definition 3.5 (d-dimensional Bernstein polynomials) For $f \in C[0,1]^d$, the d-dimensional Bernstein polynomials of f are given by the formula

$$B_n(f; x_1, \ldots, x_d) := \textstyle\sum_{k_1=0}^{n} \cdots \sum_{k_d=0}^{n} b_{n,k_1}(x_1) \cdots b_{n,k_d}(x_d) f(\frac{k_1}{n}, \ldots, \frac{k_d}{n}).$$

Theorem 3.6 (d-dimensional Weierstrass) *The restrictions of the polynomials in d arguments to any compact subset K of $\mathbb{R}^d$ is dense in $C(K)$.*

Proof. It is sufficient to prove the theorem for $K = I = [0,1]^d$. Apply Korovkin with

$$F = \{x^\alpha\} \in \mathcal{P}_{2,\ldots,2}, \quad p(x,t) = \sum_{i=1}^{d} (x_i - t_i)^2, \quad U_n = B_n. \qquad □$$

[1] Sergei Bernstein (1880-1968), Russian mathematician, one of the "fathers" of Approx. Theory, PhD from the Sorbonne in 1904, in his thesis solved Hilbert's 19th Problem (first Problem to be solved), however on returning back to Russia in 1905 he had to start his doctorate again because Russia did not recognize foreign qualification for university posts; so, he solved Hilbert's 20th Problem. In 1955 he became the third Russian elected as a foreign member of the Paris Academy of Sciences (after the tsar Peter the Great and mathematician P. Chebyshev).

3.2 Fejer sums and the second Weierstrass theorem

Definition 3.7 (Fourier[2]sums) For $f \in L_1(\mathbb{T})$, its n-th partial Fourier series is given by

$$s_n(f,x) := \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx, \qquad \begin{bmatrix} a_k \\ b_k \end{bmatrix} := \frac{1}{\pi}\int_{\mathbb{T}} f(t) \begin{bmatrix} \cos kt \\ \sin kt \end{bmatrix} dt\,.$$

Lemma 3.8 *One has*

$$s_n(f,x) = \frac{1}{\pi}\int_{\mathbb{T}} D_n(x-t)f(t)dt, \quad D_n(x) = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{1}{2}x} \quad \text{– the Dirichlet kernel.}$$

Proof. From definition

$$a_k \cos kx + b_k \sin kx = \frac{1}{\pi}\int_{\mathbb{T}} f(t)\left[\cos kt \cos kx + \sin kt \sin kx\right] dt = \frac{1}{\pi}\int_{\mathbb{T}} f(t) \cos k(x-t)\, dt\,,$$

so that

$$s_n(f,x) = \frac{1}{\pi}\int_{\mathbb{T}} D_n(x-t)f(t)\,dt, \qquad D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx\,.$$

It remains to find the expression for the sum. Since $2\sin\frac{1}{2}x\cos kx = \sin(k+\frac{1}{2})x - \sin(k-\frac{1}{2})x$, we see that

$$2\sin\tfrac{1}{2}x \cdot D_n(x) = \sin\tfrac{1}{2}x + [\sin\tfrac{3}{2}x - \sin\tfrac{1}{2}x] + \cdots + [\sin(n+\tfrac{1}{2})x - \sin(n-\tfrac{1}{2})x] = \sin(n+\tfrac{1}{2})x,$$

hence the result. □

Definition 3.9 (Fejer[3]sums [1904]) The Fejer operator associates with f the average of its partial Fourier series of orders up to $n-1$,

$$\sigma_n(f) = \frac{1}{n}\sum_{j=0}^{n-1} s_j(f).$$

In the same way,

$$\sigma_n(f,x) = \frac{1}{\pi}\int_{\mathbb{T}} F_n(x-t)f(t)dt, \quad F_n(x) = \frac{1}{n}\frac{\sin^2\frac{n}{2}x}{2\sin^2\frac{1}{2}x} \quad \text{– the Fejer kernel.} \tag{3.3}$$

Lemma 3.10 *The Fejer operator σ_n is a positive operator (which s_n is not), and*

$$\sigma_n\left(\begin{matrix}\cos kt \\ \sin kt\end{matrix}, x\right) = \left(1 - \frac{k}{n}\right) \begin{matrix}\cos kx \\ \sin kx\end{matrix}$$

hence $\sigma_n \to 1$ on $\mathcal{T}_k$ for any k, in particular, $\sigma_n(p_i) \to p_i$ for $p_i(x) \in \{1, \cos x, \sin x\}$.

Theorem 3.11 (Weierstrass [1885]) *The set $\mathcal{T}$ of all trigonometric polynomial is dense in $C(\mathbb{T})$.*

Proof. For $\mathbb{T} = [-\pi, \pi)$, the Korovkin Theorem, with the choices

$$\mathcal{F} = \{1, \cos x, \sin x\} \in \mathcal{T}_1, \quad p(x,t) = 1 - \cos(x-t), \quad U_n = \sigma_n,$$

shows that $\sigma_n(f) \to f$ for all $f \in C(\mathbb{T})$. □

Remark 3.12 Notice that, since $\|s_n\| \sim \ln n$, the Fourier sums (s_n) fail to converge to 1 on $C(\mathbb{T})$.

[2]Jean Baptiste Fourier, 1768-1830, French mathematician and politician, in 1798 participated in Napoleon's invasion of Egypt (as a scientific advisor), in 1801 was appointed by Napoleon as the Prefect of Grenobl, the post he held till 1813, it was during that time when he did his important work on the theory of heat.

[3]Lipot Fejer, 1880-1959, was born Leopold Weiss but changed his name around 1900 to make himself more Hungarian, studium in Budapest and Berlin, since 1911 the chair of mathematics at the University of Budapest.

3.3 Exercises

3.1. Prove: If $f \in C[0,1]$ vanishes at 0 and 1, then the sequence $(B_n^* f)$,

$$B_n^*(f,x) := \sum_{k=1}^{n-1} \left\lfloor \binom{n}{k} f\left(\tfrac{k}{n}\right) \right\rfloor x^k (1-x)^{n-k},$$

which consists of polynomials with *integer* coefficients in the standard *power* form, converges uniformly to f. (Here $\lfloor t \rfloor$ is the largest integer not bigger than t.)

Hint. Show that $|B_n(f,x) - B_n^*(f,x)| \le \frac{1}{n}$.

3.2. Prove: If $f \in C^k[0,1]$, then not only does $B_n(f)$ converge uniformly to f, but also the k-th derivatives of $B_n(f)$ converge uniformly to $f^{(k)}$.

Hint. From (3.2) derive an expression for $B_n^{(k)}(f)$ and compare it with $B_{n-k}(f^{(k)})$. You may use the fact that $h^{-k}\Delta_h^k(f,t) = f^{(k)}(\xi)$ with some $\xi \in [t, t+kh]$.

3.3. Find explicit expression for $B_n(p_1)$, $B_n(p_2)$ and $B_n(p_3)$, where $p_i(x) = x^i$, directly from the formula (3.1). *Hint.* Differentiate the identity

$$\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = (p+q)^n$$

with respect to p, and proceed by induction.

3.4. For the Fejer kernel F_n, prove the formula (3.3), and that $\|F_n\|_{L_1[-\pi,\pi]} = \pi$. Prove that the Fejer operator $\sigma_n : C(\mathbb{T}) \to C(\mathbb{T})$ satisfies $\|\sigma_n\| = 1$. Why is the Lebesgue inequality

$$\|f - \sigma_n(f)\| \le (\|\sigma_n\| + 1) E_n(f) = 2E_n(f)$$

not applicable to the linear operator σ_n?