4 Existence and uniqueness of best approximation

4.1 Existence

**Lemma 4.1** Let \( U \) be a compact set in a metric space \( \mathcal{X} \). Then, for every \( f \) in \( \mathcal{X} \), there exists an element of best approximation.

**Proof.** Let \( d_\ast := \inf \{ d(f, u) : u \in U \} \), and let \((u_i)\) be a minimizing sequence, i.e., \( d(f, u_i) \to d_\ast \). By the compactness of \( U \), this sequence has at least one limit point \( u_\ast \in U \), and we may assume that \( d(u_i, u_\ast) \to 0 \). By the triangle inequality,

\[
d(f, u_\ast) \leq d(f, u_i) + d(u_i, u_\ast) \to d_\ast.
\]

**Theorem 4.2** Let \( U \) be a finite-dimensional subspace of a normed linear space \( \mathcal{X} \). Then, for every \( f \) in \( \mathcal{X} \), there exists an element of best approximation.

**Proof.** Let \( u_0 \) be any element of \( U \), e.g., \( u_0 = 0 \). The best approximant we seek lies in the set

\[
U_0 := \{ u : u \in U, \| f - u \| \leq \| f - u_0 \| \}.
\]

This set is compact because it is a closed and bounded subset of a finite-dimensional space. Therefore, by the previous theorem, there is an element \( u_\ast \) of best approximation from \( U_0 \) to \( f \).

4.2 Example of nonexistence

The finite-dimensionality hypothesis cannot be dropped as the following example shows.

Let \( c_0 \) be the Banach space of infinite sequences \( f \) such that

\[
f = (\xi_1, \xi_2, \ldots), \quad \xi_k \to 0, \quad \| f \| = \max |\xi_k|,
\]

and let \( U_0 := \{ u \in c_0 : \sum_{k=1}^{\infty} 2^{-k} \alpha_k = 0 \} \).

**Lemma 4.3** For any \( f \in c_0 \setminus U_0 \), the element of best approximation from \( U_0 \) to \( f \) does not exist.

**Proof.** Let \( f \in c_0 \setminus U_0 \) and let \( \lambda := \sum_{k=1}^{\infty} 2^{-k} \xi_k \neq 0 \).

1) On the one hand, the following elements belong to \( U_0 \):

\[
u_1 = f - \frac{2}{3} \lambda (1, 0, 0, \ldots), \quad u_2 = f - \frac{2}{3} \lambda (1, 1, 0, 0, \ldots), \quad u_3 = f - \frac{2}{3} \lambda (1, 1, 1, 0, 0, \ldots), \quad \text{etc.,}
\]

and \( \| f - u_n \| = (1 - \frac{1}{2^n})^{-1} |\lambda| \setminus |\lambda| \). Hence, \( d(f, U_0) \leq |\lambda| \).

2) On the other hand, for any \( u \in U_0 \), we have \( \| f - u \| > |\lambda| \) because

\[
|\lambda| = |\sum_{k=1}^{\infty} 2^{-k} \xi_k| = |\sum_{k=1}^{\infty} 2^{-k} (\xi_k - \alpha_k)| \leq \sum_{k=1}^{\infty} 2^{-k} |\xi_k - \alpha_k| < \| f - u \| \sum_{k=1}^{\infty} 2^{-k} = \| f - u \|,
\]

the last inequality being strict since \( \xi_k, \alpha_k \to 0 \) implies \( |\xi_k - \alpha_k| < \| f - u \| \) for \( k > k_0 \).

3) Thus, for any \( f \in c_0 \setminus U_0 \), one has \( d(f, U_0) = |\lambda(f)| \), but the element of best approximation does not exist.
4.3 Uniqueness

Definition 4.4 A normed linear space $X$ is called strictly convex if the unit sphere contains no line segments on its surface, i.e.,
\[ \|x\| = \|y\| = 1, \quad x \neq y \Rightarrow \|\frac{1}{2}(x + y)\| < \frac{1}{2}\|x\| + \frac{1}{2}\|y\| = 1 \quad \forall x, y. \]

Lemma 4.5 Let $U$ be a subspace of a strictly convex normed linear space $X$. Then, for each element $f \in X$, there is at most one element of best approximation.

Proof. Suppose that $u_1$ and $u_2$ are two different best approximations from $U$ to $f$ and $\|f - u_i\| = \lambda_i$. Then
\[ \|f - \frac{1}{2}(u_1 + u_2)\| = \|\frac{1}{2}(f - u_1) + \frac{1}{2}(f - u_2)\| < \frac{1}{2}\|f - u_1\| + \frac{1}{2}\|f - u_2\| = \lambda, \]
a contradiction to the definition of best approximation.

Theorem 4.6 Let $U_n$ be a finite-dimensional subspace of a strictly convex normed linear space $X$. Then, for each $f \in X$, there exists a unique element of best approximation.

Example 4.7 Any Hilbert space with a scalar product $(f, g)$ and the norm $\|f\| := (f, f)^{1/2}$ is strictly convex. This follows from the identity
\[ \|\frac{1}{2}(f + g)\|^2 + \|\frac{1}{2}(f - g)\|^2 = \frac{1}{2}\|f\|^2 + \frac{1}{2}\|g\|^2. \]

Example 4.8 The spaces $L_p[a, b]$ with the norm $\|f\| := (\int_a^b |f(t)|^p dt)^{1/p}$ are strictly convex if $1 < p < \infty$.

Example 4.9 The spaces $L_1[a, b]$ and $L_\infty[a, b] := C[a, b]$ are not strictly convex and examples of some subspaces which provide several b.a.’s to $X$. For example, let $X = L_1[-1, 1]$, $f(x) = \text{sgn} x, U_1 = \{0\}$. Then any $\alpha \in [-1, 1]$ is a b.a. to $f$.

However, the nonuniqueness is not guaranteed. If in (a) one takes $f(x) = \text{sgn}(x - x_0)$ with any $x_0 \neq 0$, then a.b. from $U_1$ is unique.

4.4 Example of nonuniqueness in $L_1$

Lemma 4.10 1) Let $f \in L_1(\mathbb{T})$. Then, for $0 < k < n$, we have $\int_0^\pi f(nx) e^{imx} dx = 0$

2) If also $f \perp 1$, then $f(x) \perp T_{n-1}$.

Proof. Since $e^{imx}$ are linear combination of $e^{imx}$ with $0 < |m| < n$, it will be enough to prove that $I := \int_0^\pi f(nx) e^{imx} dx = 0$. Setting $x = t + \frac{\pi}{n}$, and using the equality $\int_T g = \int_{T+\alpha} g$, we obtain
\[ I = \int_{-\pi}^\pi f(nx) e^{imx} dx = \int_{-\pi - 2\pi/n}^{\pi - 2\pi/n} f(nt + \frac{\pi}{n}) e^{imt+2im\pi/n} dt = e^{2im\pi/n} \int_{-\pi}^\pi f(nt) e^{imt} dt = e^{2im\pi/n} I. \]
Since $0 < |m| < n$, we have $e^{2im\pi/n} \neq 1$, hence $I = 0$.

Lemma 4.11 For $F(x) = \text{sgn} \sin nx$, any $s \in T_{n-1}$ with $\|s\|_\infty < 1$ is a polynomial of b.a. to $F$ in $L_1(\mathbb{T})$.

Proof. By previous lemma, $F \perp T_{n-1}$. Then, for any $s \in T_{n-1}$,
\[ \int_T |F(x) - s(x)| dx \geq \int_T |F(x) - s(x)| \text{sgn} \sin nx dx = \int_T F(x) \text{sgn} \sin nx dx = 2\pi. \]
If $\|s\|_\infty < 1 = \|F\|$, then $\text{sgn} (F(x) - s(x)) = \text{sgn} F(x) = \text{sgn} \sin nx$, and the first inequality becomes equality.

This example has a remarkable generalization.

Theorem 4.12 (Hobby–Rice) For any $n$-dimensional subspace $U_n$ of $L_1[a, b]$ there exists a sign function $h$ with $n$ breakpoints such that $h \perp U_n$.

Theorem 4.13 (Krein[1938]) No finite-dimensional subspace of $L_1[a, b]$ is a unicity space, i.e., for any $U_n$ in $L_1[a, b]$ there exists a function $f \in L_1[a, b]$ that has several b.a.

1Mark Krein (1907-1989), Russian mathematician, one of the recipients of prestigious Wolf Prize in mathematics “for his fundamental contributions to functional analysis and its applications”.
4.5 Exercises

4.1. Prove that the equality \( \|x\| = \|y\| = \|\frac{1}{2}x + \frac{1}{2}y\| = 1 \) implies that \( \|\alpha x + (1 - \alpha)y\| = 1 \) for any \( \alpha \in [0, 1] \), i.e., that two conditions in Definition 4.4 are really equivalent.

4.2. Does the unicity Theorem 4.6 have a converse? That is, can we infer strict convexity from a knowledge that for each \( f \in X \) and for each finite-dimensional \( U \subset X \) there exists a unique element of best approximation?

*Hint.* Let \( X = \mathbb{R}^2 \) with a norm which is not strictly convex, i.e., the set \( \|x\| = 1 \) contains a line segment. Find \( f \) and one-dimensional subspace \( U \) with several b.a. to \( f \).

4.3. Fill in the details of the following proof of Theorem 4.2. It is based on the Weierstrass theorem: A continuous function on a closed bounded set in \( \mathbb{R}^n \) attains its minimum.

\( a) \) Let \( f \in X \), an let \( U_n = \text{span} (g_k)_{k=1}^n \) where \( g_k \) are linearly independent.

\( b) \) For \( x = (x_1, \ldots, x_n) \), the function \( G(x) := \| \sum_{k=1}^n x_k g_k \| \) is a continuous function of \( n \) real variables \( x_k \), hence, on the unit ball \( \sum x_k^2 = 1 \), it takes some minimal value \( \rho > 0 \).

\( c) \) The function \( F(x) := \| f - \sum_{k=1}^n x_k g_k \| \) is continuous as well, and it follows from (b) that \( F(x) \to \infty \) as \( x \to \infty \).

\( d) \) Thus, looking for the infimum of \( F(x) \) over \( x \in \mathbb{R}^n \), we may restrict ourselves to some ball \( B_r := \{ x : \sum x_k^2 \leq r \} \), this is a bounded closed set in \( \mathbb{R}^n \), hence, on \( B_r \), \( F(x) \) attains its minimal value.

4.4. The essence of the example given in Lemma 4.3 is that the linear functional

\[ \lambda : c_0 \to \mathbb{R}, \quad \lambda(f) := \sum_{k=1}^{\infty} 2^{-k} \xi_k, \quad f = (\xi_1, \xi_2, \ldots) \in c_0 \]

does not take its norm

\[ \|\lambda\| := \sup \{ |\lambda(f)| : \|f\| = 1 \}, \]

i.e., there is no function \( f \in c_0 \) such that \( |\lambda(f)| = \|\lambda\| \|f\| \).

\( a) \) Prove that, for any \( X \) and any \( \lambda \in X^* \) (the space of all linear bounded functionals on \( X \)), we have the equality \( |\lambda(f)| = \|\lambda\| \text{dist}(f, \ker \lambda) \) for any \( f \in X \).

\( b) \) Using (a), even if you have not proved it, prove that if \( \lambda : X \to \mathbb{R} \) does not take its norm, then the subspace \( U_0 := \ker \lambda \) is a non-existence set.

\( c) \) For \( \ell_1 \), the Banach space of infinite sequences \( f \) such that

\[ f = (\xi_1, \xi_2, \ldots), \quad \sum_{k=1}^{\infty} |\xi_k| < \infty, \quad \|f\| = \sum_k |\xi_k|, \]

construct a functional \( \lambda(f) = \sum_{k=1}^{\infty} \lambda_k \xi_k \) which does not take its norm, thus find a subspace of \( \ell_1 \) which is a non-existence set.


*Hint.* For any \( f \in U_n \), consider the function \( F = |f| \cdot h \), where \( h \) is a sign function from Hobby–Rice theorem.