Approximation Theory – Lecture 5

5 Best approximation in $C[a,b]$

5.1 Characterization

Theorem 5.1 (Kolmogorov[1][1948]) *Let $\mathcal{U}$ be a linear subspace of $C(K)$. An element $p^* \in \mathcal{U}$ is a best approximation to $f \in C(K)$ if and only if*

$$\max_{x\in\mathcal{Z}} [f(x) - p^*(x)]\, q(x) \ge 0 \quad \forall q \in \mathcal{U}\,, \tag{5.1}$$

where $\mathcal{Z}$ is the set of all points for which $|f(x)-p^*(x)| = \|f-p^*\|$.

Proof. 1) Suppose that condition (5.1) is fulfilled. Take any $p \in \mathcal{U}$ and set $q = p^* - p$. We see that there is a point $x_0 \in \mathcal{Z}$ such that $[f(x_0) - p^*(x_0)]\, q(x_0) \ge 0$ Then

$$\begin{aligned} |f(x_0) - p(x_0)|^2 &= |f(x_0) - p^*(x_0) + q(x_0)|^2 \\ &= |f(x_0) - p^*(x_0)|^2 + 2[f(x_0) - p^*(x_0)]\, q(x_0) + |q(x_0)|^2 \\ &\ge |f(x_0) - p^*(x_0)|^2 = \|f - p^*\|^2, \end{aligned}$$

i.e., any $p \in \mathcal{U}$ approximates f not better than p^*.

2) Conversely, if (5.1) is not true, then there exists a q such that

$$\max_{x\in\mathcal{Z}} [f(x) - p^*(x)]\, q(x) = -2\varepsilon$$

for some $\varepsilon > 0$. By continuity, there is an open subset G of K such that

$$G \supset \mathcal{Z}, \quad [f(x) - p^*(x)]\, q(x) < -\varepsilon \quad \text{on} \quad G.$$

Let us see how f is approximated by $p := p^* - \lambda q$. Set $m = \|q\|$ and $E(f) = \|f - p^*\|$.

a) For $x \in G$, we have

$$\begin{aligned} |f(x) - p(x)|^2 &= |f(x) - p^*(x) + \lambda q(x)|^2 \\ &= |f(x) - p^*(x)|^2 + 2\lambda[f(x) - p^*(x)]\, q(x) + \lambda^2 |q(x)|^2 \\ &< E(f)^2 - 2\lambda\varepsilon + \lambda^2 m^2. \end{aligned}$$

If we take $\lambda < \varepsilon/m^2$, then we obtain that $|f(x) - p(x)|^2 < E(f)^2 - \lambda\varepsilon$ on G.

b) On the complement $F = K \setminus G$, we have $|f(x) - p^*(x)| < E(f)$. But F is closed, hence there is also a $\delta > 0$ such that $|f(x) - p^*(x)| \le E(f) - 2\delta$ on F. Taking $\lambda < \delta/m$, we obtain

$$|f(x) - p(x)| \le |f(x) - p^*(x)| + \lambda|q(x)| \le E(f) - 2\delta + \lambda m < E(f) - \delta \quad \text{on} \quad F\,. \qquad \square$$

Example 5.2 For $f \in C[a,b]$ and $\mathcal{U} = \mathcal{P}_0$ (the constant functions), the best approximation is given by

$$p^*(x) = \tfrac{1}{2}\,(\max f + \min f).$$

[1] Andrey Kolmogorov (or, Kolmogoroff), 1903-1987, Russian mathematician, made fundamental contributions to probability, topology, functional analysis, mechanics, etc., etc. In short: one of the greatest mathematician of the 20-th century.

5.2 Chebyshev alternation theorem

Theorem 5.3 (Chebyshev[2][1854]) *A polynomial $p^* \in \mathcal{P}_n$ is the best approximant to $f \in C[a,b]$ if and only if there exist $(n+2)$ points $a \le t_1 < \cdots < t_{n+2} \le b$ such that*

$$f(t_i) - p^*(t_i) = (-1)^i \gamma, \qquad |\gamma| = \|f - p^*\|, \tag{5.2}$$

i.e., if and only if the difference $f(x) - p^*(x)$ takes consecutively its maximal value with alternating signs at least $(n+2)$ times.

Proof. 1) Assume that the difference $f - p^*$ takes the value $\|f - p^*\|$ with alternating signs in $m \le n+1$ points. Set

$$\mathcal{Z} = \{x \in [a,b] : |f(x) - p^*(x)| = \|f - p^*\|\}, \quad \mathcal{Z}_\pm = \{x \in [a,b] : f(x) - p^*(x) = \pm\|f - p^*\|\}.$$

Then there exists an ordered set of m disjoint intervals (K_i) which contains both $\mathcal{Z}_-$ and $\mathcal{Z}_+$ and such that, on adjacent intervals, the points from $\mathcal{Z}$ belongs to $\mathcal{Z}_-$ and $\mathcal{Z}_+$ alternatively. Select any points z_k between adjacent sets K_i:

$$K_1 < z_1 < K_2 < \cdots < z_{m-1} < K_m \,,$$

and take the polynomial

$$q(x) = \textstyle\prod_{i=1}^{m-1} (x - z_i), \quad m-1 \le n.$$

This q is in $\mathcal{P}_n$, it alternates in sign on adjacent K_is, and for this q (or for $-q$) we obtain

$$[f(x) - p^*(x)]\, q(x) < 0 \quad \text{on} \quad \mathcal{Z},$$

so that, by Theorem 5.1, p^* is not a best approximant.

2) If p^* is a polynomial that satisfies (5.2), then, for any $q \in \mathcal{P}_n$, the condition

$$[f(t_i) - p^*(t_i)]\, q(t_i) < 0, \quad i = 1, \ldots, n+2,$$

would force q to change its sign in $(n+2)$ points, hence to have $n+1$ zero, which is impossible. Thus

$$\max_{x \in (t_i)} [f(x) - p^*(x)]\, q(x) \ge 0, \quad \forall q \in \mathcal{P}_n,$$

so that, by Kolmogorov Theorem, p^* is a best approximant. □

Theorem 5.4 *A polynomial $p^* \in \mathcal{T}_n$ is the best approximant to $f \in C(\mathbb{T})$ if and only if there exist $(2n+2)$ points $-\pi < t_1 < \cdots < t_{2n+2} \le \pi$ such that*

$$f(t_i) - p^*(t_i) = (-1)^i \gamma, \qquad |\gamma| = \|f - p^*\| \,.$$

Proof. The proof is essentially the same. 1) We notice that, due to periodicity, the difference $|f - p|$ can take its maximal value with alternating signs only even number of times, say $2m$. Therefore, if $2m \le 2n$, there are $2m$ points z_k such that

$$K_1 < z_1 < K_2 < \cdots < z_{2m-1} < K_{2m} < z_{2m} < K_1 + 2\pi \,.$$

The polynomial $q \in \mathcal{T}_m$ (that violates the Kolmogorov criteria) is then defined as

$$q(x) = \prod_{i=1}^{m} \sin\frac{x - z_{2i-1}}{2} \sin\frac{x - z_{2i}}{2} = \prod_{i=1}^{m} (a_i + \cos(x - b_i)) \,.$$

2) For sufficiency, we use the fact that any $q \in \mathcal{T}_n$ can have no more than $2n$ zeros on the period. □

Remark 5.5 For $\mathcal{U} = \mathcal{P}_n$ or $\mathcal{T}_n$, the number of points required in alternation theorem is equal to $\dim(\mathcal{U}) + 1$.

[2] Pafnutij Chebyshev (or Tchebycheff, or Tschebyshev, or Chebyshov, or …), 1821-1894, great Russian mathematician, got his international fame by proving the Bertrand postulate: there is always a prime between n and $2n$, almost proved the prime number theorem showing that $\lim_{x\to\infty} \pi(x) \frac{\ln x}{x} = 1$ if the limit exists.

5.3 Exercises

5.1. For the complex valued continuous functions, Kolmogorov criterion takes the form

$$p^* \text{ is a b.a. to } f \quad \Leftrightarrow \quad \max_{x \in \mathcal{Z}} \operatorname{Re}\,[f(x) - p^*(x)]\,\overline{q(x)} \ge 0 \quad \forall q \in \mathcal{U}\,.$$

Check the "real" proof to find the changes.

5.2. Apply Kolmogorov criterion to prove the following statements:

(a)	$K = [-1,1]$,	$f(x) = x^3$,	$\mathcal{U} = \operatorname{span}\{1, x^2\}$	$\Rightarrow$	$p^*(x) = (1 - x^2)$;
(b)	$K = [-1,1]^2$,	$f(x,y) = xy$,	$\mathcal{U} = \operatorname{span}\{1, x, y\}$	$\Rightarrow$	$p^* \equiv 0$;
(c)	$K = \{\lvert x\rvert + \lvert y\rvert \le 1\}$,	$f(x,y) = x^2 + y^2$,	$\mathcal{U} = \operatorname{span}\{1, x, y, xy\}$	$\Rightarrow$	$p^*(x,y) = \frac{1}{2} + 2xy$;
(d)	$K = \{z \in \mathbb{C} : \lvert z\rvert \le 1\}$,	$f(z) = z^n$,	$\mathcal{U} = \mathcal{P}_{n-1}$	$\Rightarrow$	$p^* \equiv 0$.

For the cases (a) and (c), find another b.a. to f.

5.3. For $\mathcal{T}_{n-1}$, the space of trig. polynomials of degree $n-1$, find the value and the polynomial of best approximation to $f(x) = a\cos nx + b \sin nx$.

5.4. Let

$$f(x) = \sum_{k=0}^{\infty} a_k \cos 3^k x, \quad a_k > 0, \quad \sum_{k=0}^{\infty} a_k < \infty.$$

Prove that, for $3^m \le n < 3^{m+1}$, the polynomial

$$t_n(x) = s_{3^m}(f, x) = \sum_{k=0}^{m} a_k \cos 3^k x$$

(which is a partial Fourier series to f) is the b.a. to f from $\mathcal{T}_n$ and find the value of $E_n(f)$.

5.5. Use the previous result to prove the *lethargy theorem*: For any sequence $\varepsilon_n \searrow 0$, there exists an $f \in C(\mathbb{T})$ such that $E_n(f) \ge \varepsilon_n$ all n. (*Hint.* Set $a_k := \varepsilon_{k-1} - \varepsilon_k$.)

5.6. Prove de La Valle-Poussin theorem: If $p \in \mathcal{P}_n$ is a polynomial such that

$$f(t_i) - p(t_i) = (-1)^i \varepsilon_i, \qquad \operatorname{sgn} \varepsilon_i = \text{const},$$

at $n+2$ consecutive points, then $E_n(f) \ge \min_i |\varepsilon_i|$.

Use this theorem to establish the sufficiency half of the alternation theorem.

5.7. Let $f \in C[a,b]$, and let $p, q \in \mathcal{P}_{n+1}$ be alg. polinomials of degree $n+1$ determined by the condition

$$p(t_i) = f(t_i), \quad q(t_i) = (-1)^i, \qquad \text{where} \quad a \le t_1 < \cdots < t_{n+2} \le b.$$

Prove that the polynomial $r := p - \lambda q$, where λ is chosen so that r belongs to $\mathcal{P}_n$, is the best approximation to f on (t_i) and find the error.

Remark. This is a practical method for constructing the b.a. on the set of $n+2$ points.