Part III - Lent Term 2005
Approximation Theory – Lecture 6

6 Best approximation in $C[a,b]$ (cont.)

6.1 Chebyshev systems

The proof of the alternation theorem is based on the following two properties of $\mathcal{P}_n$:
1) For any set of m distinct points (z_i), where $m \le n$, there exists a polynomial $q \in \mathcal{P}_n$ that changes its sign exactly at these points;
2) Any $p \in \mathcal{P}_n$ has at most n distinct zeros in $[a,b]$.

So, it is suggestive to make some generalizations.

Definition 6.1 (Chebyshev systems) A set $\Phi = (u_0, \dots, u_n)$ from $C(K)$ is a *Chebyshev* system, if it satisfies the *Haar*[1]condition: each polynomial

$$p = a_0 u_0 + \cdots + a_n u_n,$$

with not all coefficients equal to zero, has at most n distinct zeros on K. The $(n+1)$-dimensional space $\mathcal{U}_n$ spanned by such a Φ is called a *Chebyshev space*.

Lemma 6.2 *The following conditions are equivalent.*
(i) $(u_i)_0^n$ *is a Chebyshev system.*
(ii) *For any $n+1$ distinct points $(x_i)_0^n \in K$, the following determinant is not zero:*

$$D(x_0, x_1, \dots, x_n) := \begin{vmatrix} u_0(x_0) & \cdots & u_n(x_0) \\ \cdots & \cdots & \cdots \\ u_0(x_n) & \cdots & u_n(x_n) \end{vmatrix}. \tag{6.1}$$

(iii) *If $(x_i)_0^n$ are distinct points of K and $(y_i)_0^n$ are arbitrary numbers, then the interpolation problem*

$$a_0 u_0(x_i) + \cdots a_n u_n(x_i) = y_i, \quad i = 0..n,$$

has a unique solution for the unknonwns (a_j).

Proof. Condition of Definition 6.1 can be expressed as follows. If $(x_0, \dots, x_n)$ are distinct points of K, then the system of $n+1$ equations with $n+1$ unknowns a_js

$$a_0 u_0(x_i) + \cdots a_n u_n(x_i) = 0, \quad i = 0..n,$$

has only the trivial solution $a_j = 0$. This is well-known to be equivalent to (ii) and (iii). $\square$

Lemma 6.3 *For any m distinct points $(z_i)_1^m$ in (a,b), with $m \le n$, there is a polynomial $q \in \mathcal{U}_n$ that vanishes exactly at these points (except perhaps the end-points) and changes sign at each of these points.*

Proof. For $m = n$ we take $q(x) := D(x, z_1, \dots, z_n)$ (see (6.1)). For $m < n$, we take a certain combination of similar determinants. $\square$

Theorem 6.4 *Let $\mathcal{U}_n$ be an $(n+1)$-dimemsional Chebyshev subspace of $C[a,b]$. Then $p^* \in \mathcal{U}_n$ is the best approximant to $f \in C[a,b]$ if and only if there exist $(n+2)$ points $a \le t_1 < \cdots < t_{n+2} \le b$ such that*

$$f(t_i) - p^*(t_i) = (-1)^i \gamma, \qquad |\gamma| = \|f - p^*\|, \tag{6.2}$$

i.e., if and only if the difference $f(x) - p^*(x)$ takes consecutively its maximal value with alternating signs at least $(n+2)$ times.

[1] Alfred Haar, 1885-1933, Hungarian mathematician, studied in Göttingen by Hilbert.

6.2 Haar's unicity theorem

Lemma 6.5 *Let K contains at least $(n+2)$ points. If p^* is a best approximation to $f \in C(K)$ from a Chebyshev subspace $\mathcal{U}_n$, then the set $\mathcal{Z}$ of all points for which $|f(x)-p^*(x)| = \|f-p^*\|$ contains at least $(n+2)$ points.*

Remark 6.6 This lemma is *not* covered by the Chebyshev alternation theorem. The latter is valid only for the real-valued functions on an interval $[a,b]$ or a circle $\mathbb{T}$, while the lemma is applicable to the complex-valued functions as well.

Proof. Suppose that $\mathcal{Z} = (x_i)_1^m$, where $m \le n+1$. Then, by Lemma 6.2 (iii), we can find a polynomial q such that $q(x_i) = -[f(x_i) - p^*(x_i)]$, all i, so that

$$\max_{x \in \mathcal{Z}} [f(x) - p^*(x)]\, q(x) = \max_{1 \le i \le m} \{-|f(x_i) - p^*(x_i)|^2\} < 0,$$

a contradiction to Kolmogorov criterion. $\square$

Theorem 6.7 *Let $\mathcal{U}_n$ be a Chebyshev subspace of $C(K)$. Then each $f \in C(K)$ possesses a unique polynomial of best approximation.*

Proof. Assume that, for a function f, there are two polynomials of best approximation, p and q: $\|f-p\| = \|f-q\| = E(f)$. By the triangle inequality, we see easily that also $r := \frac{1}{2}(p+q)$ is a best approximation. By the previous lemma, there are at least $(n+2)$ points for which $|f(x) - r(x)| = E(f)$. At each such point x, for the real numbers

$$\alpha := f(x) - p(x), \quad \beta := f(x) - q(x),$$

we have $|\alpha + \beta| = 2E(f)$, $|\alpha| \le E(f)$, $|\beta| \le E(f)$. But this is possible only if $\alpha = \beta$. Thus, p and q coincide at least at $(n+2)$ points, and because of the Haar condition they are equal identically. $\square$

This theorem has a remarkable converse (given here without a proof).

Theorem 6.8 (Haar [1918]) *Let $(u_i)_0^n$ be linearly independent continuous functions on K that contains at least $(n+2)$ points. Then each $f \in C(K)$ has only one polynomial of best approximation if and only if Φ is a Chebyshev system.*

The next theorem shows, however, that all the remarkable properties of the real-valued Chebyshev systems are restricted, in a sense, to the functions given on an interval.

Theorem 6.9 (Loss of Haar[2]) *If a compact K contains a fork (a Y-shaped curve), then there is no Haar subspaces of $C(K)$ of dimension > 1.*

Proof. Indeed, we can put x_0 and x_1 on the two arms of the Y, any other points being on the trunk of the Y, and then the points can be kept apart and moved continuously so that x_0 and x_1 change places, and so that the remaining points return to their original positions. Thus the sign of the determinant of $D(x_0, x_1, \ldots, x_n)$ is reversed. It follows that a zero determinant must occur during the moves, which excludes the Haar condition. $\square$

In particular, no polynomial space of dimension > 1 in more than one variable can be Chebyshev, and that makes the construction of uniform best approximations to functions of several arguments something of an art.

[2]"Haar" in German means "hair"

6.3 Exercises

6.1. Prove that $\mathcal{T}_n$, the space of all trig. polynomials of degree $\le n$, is a Chebyshev space on $\mathbb{T} = [-\pi, \pi)$.

Hint. Write $t_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$, substitute $z = e^{ix}$, and reduce the problem to the algebraic case.

6.2. Prove that, for distinct λ_k, $\mathcal{U}_n = \mathrm{span}(e^{\lambda_k x})_{k=0}^{n}$ is a Chebyshev system on any $[a, b]$.

Hint. For $p(x) = c_0 + \sum_{k=1}^{n+1} c_k e^{\mu_k x}$, $p \in \mathcal{U}_{n+1}$, we have $p' \in \mathcal{U}_n$. Use this fact with induction.

6.3. Using arguments similar to those used in the proof of Theorem 6.7 prove that, on the circle (i.e. on the period $\mathbb{T} = [-\pi, \pi)$), there is no Chebyshev space of *even* dimension.

Remark. Chebyshev spaces on the circle of *odd* dimension do exist, e.g. $\mathcal{T}_n$.

6.4. Repeat the Haar construction for

$$\Phi = \{1, x^2\} \quad \text{on} \quad [-1, 1].$$

Find appropriate f and a set of its best approximations.

Theorem 6.8 (Haar [1918]) *Let $(u_i)_0^n$ be linearly independent continuous functions on K that contains at least $(n+2)$ points. Then each $f \in C(K)$ has only one polynomial of best approximation if and only if Φ is a Chebyshev system.*

Proof. Half of this theorem has been already established. For another half, suppose that Φ is not a Chebyshev system. Then there exist points $(x_i)_0^n$ such that the matrix $[u_i(x_j)]$ is singular. Let non-zero vectors $[a_0, \ldots, a_n]$ and $[b_0, \ldots, b_n]$ be selected orthogonal to the columns and rows, respectively, of this matrix. Thus

$$1) \quad \textstyle\sum_i a_i u_i(x_j) = 0, \quad j = 0..n; \quad \text{and} \quad 2) \quad \textstyle\sum_j b_j u_i(x_j) = 0, \quad i = 0..n.$$

Set $q := \sum_i a_i u_i$ and let $p = \sum_i c_i u_i$ be any polynomial. Then it follows that

$$1') \quad q(x_j) = 0, \quad j = 0..n, \qquad 2') \quad \textstyle\sum_j b_j p(x_j) = 0 \quad \forall\, p \in \mathcal{U}_n.$$

A) Let $\mathcal{F}$ be the class of functions $f \in C(K)$ such that $f(x_i) := \operatorname{sgn} b_i = \pm 1$ for all non-zero b_is, and $\|f\| = 1$. Then, for any $f \in \mathcal{F}$, and for any polynomial $p \in \mathcal{U}_n$, at some x_j we must have

$$\|f - p\| \ge |f(x_j) - p(x_j)| \ge 1\,.$$

Indeed, if $|f(x_j) - p(x_j)| < 1 = |f(x_j)|$ for all such x_j, then $\operatorname{sgn} p(x_j) = \operatorname{sgn} f(x_j) := \operatorname{sgn} b_j$, contradicting the equality $\sum_j b_j p(x_j) = 0$. Hence, $E_n(f) \ge 1$ for all $f \in \mathcal{F}$.

B) Now, select any $f \in \mathcal{F}$, suppose that $\|q\| \le 1$, and set

$$F(x) = (1 - |q(x)|) f(x)\,.$$

This F is also in $\mathcal{F}$, hence $E_n(F) \ge 1$. But for any $\lambda \in [0,1]$, the polynomial λq is a best approximation to F because

$$|F(x) - \lambda q(x)| \le |F(x)| + \lambda |q(x)| \le 1 - |q(x)| + |q(x)| = 1. \qquad \square$$