Part III - Lent Term 2005
Approximation Theory – Lecture 7

7 Chebyshev polynomials

7.1 Basic properties

Definition 7.1 Chebyshev polynomial of degree n is defined as

$$T_n(x) = \cos n \arccos x, \quad x \in [-1, 1],$$

or, in a more instructive form,

$$T_n(x) = \cos n\theta, \quad x = \cos\theta, \quad \theta \in [0, \pi].$$

One sees at once that, on $[-1, 1]$,
1) T_n takes its maximal value with alternating signs $(n+1)$ times:

$$\|T_n\| = 1, \quad T_n(x_k) = (-1)^k, \quad x_k = \cos\tfrac{\pi k}{n}, \quad k = 0..n,$$

2) T_n has n distinct zeros:

$$T_n(t_k) = 0, \quad t_k = \cos\tfrac{(2k-1)\pi}{2n}, \quad k = 1..n.$$

Lemma 7.2 *Chebyshev polynomials T_n satisfy the recurrence relation*

$$T_0(x) \equiv 1, \quad T_1(x) = x, \tag{7.1}$$
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \ge 1. \tag{7.2}$$

In particular, T_n is indeed an algebraic polynomial of degree n with the leading coefficient 2^{n-1}.

Proof. Expressions (7.1) are straightforward, the recurrence follows from the equality

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta$$

via substitution $x = \cos\theta$. □

Theorem 7.3 *On the interval $[-1, 1]$, among all polynomials of degree n with leading coefficient $a_n = 1$, the Chebyshev polynomial $\frac{1}{2^{n-1}}T_n$ deviates least from zero, i.e.,*

$$\inf_{(a_i)} \|x^n + a_{n-1}x^{n-1} + \cdots + a_0\| = \tfrac{1}{2^{n-1}}\|T_n\|\,.$$

Remark 7.4 Actually, we seek the value $\inf\limits_{p\in\mathcal{P}_{n-1}} \|f - p\|$ of b.a. to $f(x) = x^n$ from $\mathcal{P}_{n-1}$ in $C[-1, 1]$.

Proof. Let $f(x) = x^n$ and let $p^* \in \mathcal{P}_{n-1}$ be its best approximation in $C[-1, 1]$. Then (a) the difference $f - p^*$ is a polynomial of degree n with leading coefficient equal to 1, and (b) by Chebyshev alternation theorem, it takes its maximal value at least $(n+1)$ times with alternating signs. As we have seen, the polynomial $\frac{1}{2^{n-1}}T_n$ enjoys the same properties, hence $f - p^* = \frac{1}{2^{n-1}}T_n$. □

Corollary 7.5 *For $\Delta = \{-1 \le t_0 \le \cdots \le t_n \le 1\}$, let $\omega_\Delta(x) = \prod_{k=0}^n (x - t_k)$. For all n, we have*

$$\inf_\Delta \|\omega_\Delta\| = \tfrac{1}{2^n}\|T_{n+1}\|\,.$$

Discussion 7.6 The error of interpolation of $f \in C^{n+1}[-1, 1]$ by the Lagrange polynomial of degree n on $\Delta = (t_k)_{k=0}^n$ is given by the formula

$$f(x) - \ell_n(x) = \frac{1}{(n+1)!}\,\omega_\Delta(x)\, f^{(n+1)}(\xi), \qquad \omega_\Delta(x) = \prod_{k=0}^n (x - t_k)\,.$$

If we want to minimize the error on the class of functions f with $\|f^{(n+1)}\| \le M$, then the previous result shows that interpolation at Chebyshev knots $t_k = \cos\frac{(2k-1)\pi}{2(n+1)}$ is the optimal solution.

7.2 Estimates outside the interval

Lemma 7.7 *Let $|\tau| > 1$. Then*

$$\inf \{ \|p\| : p \in \mathcal{P}_n \,,\; p(\tau) = 1 \} = 1/|T_n(\tau)| \,.$$

Proof. Set $q(x) = T_n(x)/T_n(\tau)$, so that $q(\tau) = p(\tau) = 1$ and

$$q(x_k) = (-1)^k \gamma, \quad |\gamma| = \|q\|, \quad k = 0..n \,.$$

The inequality $\|p\| < \|q\|$ will provide for the difference $r := q - p$ the following sign pattern $\operatorname{sgn} r(x_k) = \operatorname{sgn} q(x_k) = (-1)^k$, thus it will force the polynomial $r \in \mathcal{P}_n$ to have at least n zeros on $[-1, 1]$, plus the $(n+1)$-st zero at τ, a contradiction. Hence $\|p\| \ge \|q\| = 1/|T_n(\tau)|$. □

Definition 7.8 On the interval $[a, b]$, the Chebyshev polynomial is given by

$$T_n^*(x) = T_n(y), \quad y = \tfrac{2}{b-a}\left(x - \tfrac{a+b}{2}\right)$$

Notice that its leading coefficient is equal to $2^{n-1}(\frac{2}{b-a})^n$.

Lemma 7.9 *Let $0 < a < b$. Then*

$$\inf \{ \|p_n\| : p_n(0) = 1 \} = 1/T_n^*(0) < 2\gamma^n, \quad \gamma = \tfrac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}} < 1. \tag{7.3}$$

Proof. We need only to show the estimate $2|T_n^*(0)| > 1/\gamma^n$, and that follows from the formula

$$T_n(x) = \tfrac{1}{2}\left[(x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n\right] \quad \forall\, x \in \mathbb{R}. \tag{7.4}$$

7.3 Chebyshev iterations

Discussion 7.10 For solving the system of linear equations $Ax = b$ with a positive definite matrix A one can apply the linear stationary n-step semi-iterative method

$$x^{k+i/n} = x^{k+(i-1)/n} - \tau_i(Ax^{k+(i-1)/n} - b), \quad i = 1..n \,,$$

with some $\tau_i \in \mathbb{R}$. Then the error $e^k := x^k - x^*$ satisfies

$$e^{k+1} = He^k, \quad H = \textstyle\prod_{i=1}^n (I - \tau_i A) \,,$$

and, if measured in Euclidean ℓ_2-norm, decrease of the error is

$$\|e^{k+1}\|_2 \le \rho\, \|e^k\|_2, \qquad \rho := \rho(H) = \|H\|_2 = \text{spectral radius of } H.$$

Theorem 7.11 *Let A be a positive definite matrix with the spectrum $\sigma(A) \in [m, M]$ (i.e., $m > 0$). Among all n-step semi-iterative methods, the Chebyshev method gives the largest error decrease, and we have*

$$\|e^{k+1}\|_2 \le 2\,\gamma^n \|e^k\|_2, \quad \gamma = \tfrac{\sqrt{M}-\sqrt{m}}{\sqrt{M}+\sqrt{m}} < 1.$$

Proof. Let (τ_i) be parameters of the method, and let λ_k, μ_k be the eigenvalues of A and H, respectively. Set

$$p(x) := \textstyle\prod_{i=1}^n (1 - \tau_i x).$$

Then $H = p(A)$, hence $\mu_k = p(\lambda_k)$, and we want to minimize $\rho(H) := \max_k |\mu_k|$, i.e., to minimize $\max_k |p(\lambda_k)|$ over τ_i. Since $\lambda_k \in [m, M]$, and $p(0) = 1$, this is the problem

$$p \in \mathcal{P}_n, \quad p(0) = 1, \qquad \max_{x\in[m,M]} |p(x)| \to \inf$$

which we have solved in Lemma 7.9, the solution being the Chebyshev polynomial on $[m, M]$, and the infimum value as given in (7.3). □

Remark 7.12 This result can be used for evaluating the error of the conjugate gradient method for solving $Ax = b$.

7.4 Exercises

7.1. Prove that, for any *even* trig. pol. $t_n(\theta) = \sum_{k=0}^n a_k \cos k\theta$ of degree n, the function

$$p_n(x) := t_n(\theta), \quad x = \cos\theta,$$

is an algebraic polynomial of degree n on $[-1, 1]$, and vice versa.

7.2. Derive the first (algebraic) Weierstrass theorem from the second (trigonometric) one.

7.3. Give a direct proof of Theorem 7.3, i.e. without using the Chebyshev alternation theorem, but with arguments similar to those in the proof of Lemma 7.7.

7.4. Prove formula (7.4) and derive from it the estimate (7.3).

7.5. Find the values τ_i of the Chebyshev method.

7.6. Derive from the alternation theorem (as Chebyshev did) that the polynomial $y \in \mathcal{P}_n$ of least deviation from zero satisfies the following differential equation

$$n^2(1 - y^2) = (1 - x^2)(y')^2.$$

Show that $y = T_n$ is a solution.

7.7. Prove that, at n zeros (t_i) of T_n, we have

$$T_n'(t_i) = (-1)^i \frac{n}{\sqrt{1 - t_i^2}}, \quad i = 1..n.$$

Then prove that $\|T_n\| = T_n'(1) = n^2$. Finally, using the same arguments as in the proof of Lemma 7.7 derive that

$$\text{if} \quad p \in \mathcal{P}_{n-1} \quad \text{and} \quad |p(x)| \le \frac{n}{\sqrt{1 - x^2}}, \quad \text{then} \quad \|p\| \le n^2.$$