Approximation Theory – Lecture 9

9 Degree of trigonometric approximation

9.1 Moduli of continuity

Definition 9.1 The modulus of continuity of a function $f \in C(\mathbb{T})$ is defined as

$$\omega(f,t) := \sup_{0<h<t} \sup_{x\in\mathbb{T}} |\Delta_h^1(f,x)|, \qquad \Delta_h^1(f,x) := f(x+h) - f(x).$$

Lemma 9.2 *(a)* $\omega(f,t)$ *is a continuous, non-negative and non-decreasing function of* t*;*

(b) $\omega(f+g,t) \le \omega(f,t) + \omega(g,t)$*;*

(c) $\omega(f,t) \le 2\,\|f\|$*;*

(d) $\omega(f,nt) \le n\,\omega(f,t), \quad n \in \mathbb{N}$*,*

(e) $\omega(f,t) \le t\,\|f'\|$*;*

(f) $\omega(f,\lambda t) \le (\lambda+1)\,\omega(f,t), \quad \lambda \in \mathbb{R}$*.*

Proof. (a)-(c) are straighforward, (d)-(e) follow from the properties of $\Delta_h^1(f,x)$:

$$(d') \quad \Delta_{nh}^1(f,x) = \sum_{i=0}^{n-1} \Delta_h^1(f, x+ih), \qquad (e') \quad \Delta_h^1 f(x,h) = \int_0^h f(x+u_1)\,du_1,$$

and (f) follows from (a) and (d):

$$\omega(f,\lambda t) \overset{(a)}{\le} \omega(f, \lfloor \lambda+1 \rfloor t) \overset{(d)}{\le} \lfloor \lambda+1 \rfloor \omega(f,t) \le (\lambda+1)\,\omega(f,t).$$

9.2 Direct theorems

Definition 9.3 (Convolution) Typical construction of good trigonometric approximation uses integral operators of *convolution* type

$$L_n(f,x) := \textstyle\int_{\mathbb{T}} f(x-t)K_n(t)\,dt = \int_{\mathbb{T}} f(t)K_n(x-t)\,dt, \quad K_n \in \mathcal{T}_n, \tag{9.1}$$

where K_n is a trigonometric polynomial of degree n. Since, e.g., $\cos k(x-t) = \cos kx \cos kt + \sin kx \sin kt$, the convolution $L_n(f)$ belongs to $\mathcal{T}_n$ as well.

Lemma 9.4 *Let* $K_n \in \mathcal{T}_n$ *be a polynomial that satisfies the following conditions:*

$$1)\ \textstyle\int_{\mathbb{T}} K_n(t)\,dt = 1, \qquad 2)\ K_n(t) = K_n(-t), \qquad 3)\ \int_0^\pi (nt)^k |K_n(t)|\,dt < c, \quad k = 0,1.$$

Then

$$\|f - L_n(f)\| \le c\,\omega(f, \tfrac{1}{n})\,.$$

Proof. By (1)-(3), and making use of Property (f) of $\omega(f,t)$,

$$\begin{aligned}
L_n(f,x) - f(x) &\overset{(1)}{=} \textstyle\int_{-\pi}^{\pi} [f(x-t) - f(x)]K_n(t)\,dt \\
&= \textstyle\int_0^\pi [f(x+t) - f(x)]K_n(-t)\,dt + \int_0^\pi [f(x-t) - f(x)]K_n(t)\,dt \\
&\overset{(2)}{=} \textstyle\int_0^\pi [f(x-t) - 2f(x) + f(x+t)]K_n(t)\,dt \\
&\le \textstyle\int_0^\pi 2\omega(f,t)K_n(t)\,dt = \int_0^\pi 2\omega(f, nt\frac{1}{n})K_n(t)\,dt \\
&\overset{(f)}{\le} 2\omega(f,\tfrac{1}{n}) \textstyle\int_0^\pi (nt+1)K_n(t)\,dt \\
&\overset{(3)}{\le} c\,\omega(f,\tfrac{1}{n}).
\end{aligned}$$

Definition 9.5 (Jackson kernel) The Jackson kernel J_n is given by the formula

$$m = \left\lfloor \frac{n}{2} \right\rfloor, \quad J_n(t) := \gamma_n \left(\frac{\sin mt/2}{\sin t/2} \right)^4 = \gamma_n' F_m^2(t), \quad \int_{\mathbb{T}} J_n(t)\,dt := 1,$$

where the last equality defines the constant γ_n.

Lemma 9.6 *The Jackson kernel belongs to $\mathcal{T}_n$ and, for all n,*

$$\gamma_n \sim 1/n^3, \qquad \int_0^\pi (nt)^k J_n(t)\,dt < c, \quad k = 0, 1, 2.$$

Proof. Since $t/\pi \le \sin t/2 \le t/2$ (why?), we have

$$I_{m,k} := \int_0^\pi (mt)^k \left(\frac{\sin mt/2}{\sin t/2} \right)^4 dt \sim \int_0^\pi (mt)^k \left(\frac{\sin mt/2}{t} \right)^4 dt \overset{u=\frac{mt}{2}}{\sim} m^{4-1} \int_0^{m\pi/2} u^k \left(\frac{\sin u}{u} \right)^4 du.$$

The last integral admits the lower and, for $k \le 2$, the upper estimates

$$c_k = \int_0^{\pi/2} u^k \left(\frac{2}{\pi} \right)^4 du \le \int_0^{m\pi/2} u^k \left(\frac{\sin u}{u} \right)^4 du \le \int_0^{\pi/2} u^k\,du + \int_{\pi/2}^\infty u^k \left(\frac{1}{u} \right)^4 du = c_k' + c_k'',$$

so that

$$I_{m,k} \overset{k}{\sim} m^3 \sim n^3, \quad \text{whence} \quad \int_0^\pi (nt)^k J_n(t)\,dt \overset{k}{\sim} \gamma_n I_{m,k} \overset{k}{\sim} n^3 \gamma_n.$$

For $k = 0$, the definition of γ_n implies $\gamma_n \sim n^{-3}$, hence the moment conditions for $k = 1, 2$ follow as well. □

Remark 9.7 The Fourier and the Fejer kernels do not satisfy the moment conditions (for $k = 0$ and for $k = 1$, respectively).

Theorem 9.8 (The first Jackson[1] theorem [1912]) *For some absolute constant c,*

$$E_n(f) \le c\,\omega(f, \tfrac{1}{n}) \le c\,n^{-1} \|f'\|. \tag{9.2}$$

Proof. Apply Lemma 9.4 with $K_n = J_n$. □

Theorem 9.9 (The second Jackson theorem [1912]) *If $f \in C^r(\mathbb{T})$, then*

$$E_n(f) \le c_r n^{-r} \omega(f^{(r)}, \tfrac{1}{n}) \le c_r' n^{-r} \|f^{(r)}\|.$$

Proof. Let t_n' be the polynomial of best approximation to f'. Then, by (9.2)

$$E_n(f) = E_n(f - t_n) \le c\,n^{-1} \|f' - t_n'\| = c\,n^{-1} E_n(f') \le \cdots \le c^r n^{-r} E_n(f^{(r)}),$$

hence, again by (9.2)

$$E_n(f) \le c^r n^{-r} E_n(f^{(r)}) \le c^{r+1} n^{-r} \omega(f^{(r)}, \tfrac{1}{n}).$$

What's wrong with this proof? The best approximation q_n to f' may have a constant term while t_n' does not. The following arguments fill the gap.

For any $f \in C^1(\mathbb{T})$, we have $\int_{\mathbb{T}} f'(t)\,dt = 0$. If $q_n = a_0 + \cdots$, then $\frac{1}{2\pi} \int q_n(t)\,dt = a_0$, hence $a_0 = \frac{1}{2\pi} \int [q_n(t) - f'(t)]\,dt$ and $|a_0| \le \frac{1}{2\pi} \int_{\mathbb{T}} \|q_n - f'\|\,dt = \|f' - q_n\| = E_n(f')$. Hence, for $t_n' := q_n - a_0$,

$$\|f' - t_n'\| = \|f' - (q_n - a_0)\| \le \|f' - q_n\| + \|a_0\| \le 2E_n(f'),$$

and the proof can be runned as before with additional factor 2 (at each step). □

[1] Dunham Jackson (1888-1946), American mathematician, these are results of his PhD thesis, Göttingen, advisor E. Landau.

9.3 Exercises

9.1. *Finite differences.*

For $f \in C(\mathbb{T})$ and $x \in \mathbb{T}$, the finite difference of order k with the step h is defined as

$$\Delta_h^k(f,x) = \Delta_h^1[\Delta_h^{k-1}(f,x)], \qquad \Delta_h^1 f(x) = f(x+h) - f(x).$$

Prove the following identities:

$$\begin{aligned}
&1) \quad \Delta_h^k(f,x) = \textstyle\sum_{i=0}^k (-1)^{k-i}\binom{k}{i} f(x+ih)\\
&2) \quad \Delta_{nh}^k(f,x) = \textstyle\sum_{i_1=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} \Delta_h^k(f, x+i_1 h + \cdots + i_k h),\\
&3) \quad \Delta_h^k(f,x) = \textstyle\int_0^h du_1 \int_0^h du_2 \cdots \int_0^h f^{(k)}(x+u_1+\cdots+u_k)du_k.
\end{aligned}$$

Hint. For $k=1$ they are straightforward, for $k>1$ use induction.

9.2. *Moduli of smoothness.*

The k-th modulus of smoothness of $f \in C(\mathbb{T})$ is defined by

$$\omega_k(f,t) = \sup_{0<h<t} \sup_{x\in\mathbb{T}} |\Delta_h^k(f,x)|\,.$$

Making use of Ex. 9.1 prove that

(a) $\omega_k(f,t)$ is a continuous, non-negative and non-decreasing function of t;

(b) $\omega_k(f,nt) \le n^k\,\omega_k(f,t), \quad n \in \mathbb{N}\,;$

(c) $\omega_k(f,t) \le t^k\,\|f^{(k)}\|\,;$

(d) $\omega_k(f,\lambda t) \le (\lambda+1)^k\,\omega_k(f,t), \quad \lambda \in \mathbb{R}\,.$

9.3. Prove that the Jackson operator admits the estimate in terms of the second modulus of smoothness

$$\|f - L_n(f)\| \le c\,\omega_2(f,\tfrac{1}{n}) \le c\,n^{-2}\|f''\|.$$

Hint. More generally, prove this estimate for the kernel $K_n \in \mathcal{T}_n$ that satisfies the assumptions of Lemma 9.4 with the condition (3) extended to $k=0,1,2$.

9.4. For the Fejer operator σ_n, prove the estimate

$$\|f - \sigma_n(f)\| \le c\,\omega(f,\delta_n), \quad \delta_n = \tfrac{\ln n}{n}$$

(that gives another proof of the uniform convergence $\sigma_n(f) \to f$).

Hint. Split the integral $\int_0^\pi \omega(f,t)F_n(t)\,dt = \int_0^\delta + \int_\delta^\pi$ and use properties of $\omega(f,\delta)$ together with a decay of F_n.

8.5. Prove the second equality in (9.1).