Part III - Lent Term 2005
Approximation Theory – Lecture 10

10 Degree of trigonometric approximation (cont.)

10.1 Bernstein inequality

Theorem 10.1 (Bernstein [1912]) *For any $n \in \mathbb{N}$, and for any $t_n \in \mathcal{T}_n$ on $[-\pi, \pi)$,*

$$\|t_n'\|_\infty \le n \, \|t_n\|_\infty \, . \tag{10.1}$$

Proof. We will prove stronger Szego's inequality:

$$t_n'(\xi)^2 + n^2 t_n(\xi)^2 \le n^2 \, \|t_n\|^2 \quad \forall \, \xi \in \mathbb{T}, \quad \forall \, t_n \in \mathcal{T}_n \, .$$

0) Let $t_n(\xi) = \alpha$ and let $t_n'(\xi) > 0$, say. Since $\mathcal{T}_n$ is translation-invariant and differentiation commutes with translation, it is sufficient to prove it with any particular ξ.

1) Set

$$s_n(x) := \gamma \cos nx, \quad \gamma > \|t_n\| \, ,$$

and let η be the unique point in $(-\frac{\pi}{n}, 0)$ at which $s_n(\eta) = \alpha$. Choose $\xi = \eta$, thus

$$t_n(\eta) = s_n(\eta), \quad \operatorname{sgn} t_n'(\eta) = \operatorname{sgn} s_n'(\eta) > 0.$$

Consider the trig. polynomial $q_n := s_n - t_n$. At the points $t_k := \frac{\pi k}{n}$ (extrema of $\cos nx$) q_n alternates in sign, hence it has a zero in each of $2n$ intervals (t_k, t_{k+1}), thus exactly one zero. On $(-\frac{\pi}{n}, 0)$ we have $q_n(\eta) = 0$ and $q_n(0) > 0$, so that the inequality $q_n'(\eta) \le 0$ would give us one more zero on $(\eta, 0)$. Hence, $0 < q_n'(\eta) := s_n'(\eta) - t_n'(\eta)$, i.e.,

$$0 < t_n'(\eta) < s_n'(\eta).$$

This conclusion is known as "comparison lemma".

2) Now we have

$$0 < t_n'(\eta) < s_n'(\eta) = \gamma n \sin n\eta = n\sqrt{\gamma^2 - \gamma^2 \cos^2 n\eta} = n\sqrt{\gamma^2 - t_n(\eta)^2},$$

i.e.,

$$t_n'(\eta)^2 + n^2 t_n(\eta)^2 < \gamma^2 n^2.$$

Letting $\gamma \searrow \|t_n\|$ finishes the proof. □

10.2 Inverse theorems

Throughout this section: $t_n \in \mathcal{T}_n$ is the best approximation to a given function f, and $E_n = E_n(f)$.

Theorem 10.2 (The first inverse theorem) *For any $f \in C(\mathbb{T})$, and for any $n \in \mathbb{N}$,*

$$\omega(f, \tfrac{1}{n}) \le c n^{-1} \textstyle\sum_{k=0}^n E_k(f) \, . \tag{10.2}$$

Proof. For any $\delta > 0$, and any m, we have

$$\omega(f, \delta) \le \omega(f - t_{2^m}, \delta) + \omega(t_{2^m}, \delta) \le 2E_{2^m} + \delta \, \|t_{2^m}'\| \, .$$

where we have used the properties (b), (c), (e) of the modulus of continuity from Lemma 8.2.

1) For the term E_{2^m}, due to monotone decrease of E_k, we write a trivial bound

$$E_{2^m} = 2^{-m} \textstyle\sum_{k=1}^{2^m} E_{2^m} \le 2^{-m} \sum_{k=1}^{2^m} E_k.$$

2) The bound for $\|t'_{2^m}\|$ will depend upon the relations

$$\|t_{n+k} - t_n\| \le 2E_n, \qquad 2^{s-1}E_{2^s} \le \textstyle\sum_{k=2^{s-1}+1}^{2^s} E_k,$$

and the Bernstein inequality (10.1). So,

$$\begin{aligned}
\|t'_{2^m}\| = \|t'_{2^m} - t'_0\| &\le \|t'_2 - t'_0\| + \textstyle\sum_{s=1}^{m-1} \|t'_{2^{s+1}} - t'_{2^s}\| \\
&\le 2\,\|t_2 - t_0\| + \textstyle\sum_{s=1}^{m-1} 2^{s+1}\|t_{2^{s+1}} - t_{2^s}\| \\
&\le 4E_0 + 8\textstyle\sum_{s=1}^{m-1} 2^{s-1}E_{2^s} \\
&\le 4E_0 + 8\textstyle\sum_{k=2}^{2^{m-1}} E_k \qquad \le \quad 8\textstyle\sum_{k=0}^{2^{m-1}} E_k\,.
\end{aligned}$$

Thus,

$$\omega(f,\delta) \le 8(\delta + 2^{-m})\textstyle\sum_{k=0}^{2^m} E_k(f).$$

With $\delta = \frac{1}{n}$, and m such that $2^m \le n < 2^{m+1}$, the right-hand side is $\le 8(\frac{1}{n} + \frac{2}{n})\sum_{k=0}^{n} E_k(f)$. □

Theorem 10.3 *Let $f \in C(\mathbb{T})$ and, for some $r \in \mathbb{N}$, let $\sum_{k=1}^{\infty} k^{r-1}E_k(f) < \infty$. Then $f \in C^r(\mathbb{T})$ and*

$$E_n(f^{(r)}) \le c_r\Big[n^r E_n(f) + \sum_{k=n+1}^{\infty} k^{r-1}E_k(f)\Big]\,. \tag{10.3}$$

This theorem is quite remarkable since it says that even if f is very smooth *except on a small subinterval of* $\mathbb{T}$, it will be hard to approximate f well by trig. pols.

Proof. We will use the estimate

$$(n2^{s-1})^r E_{n2^s} = n2^{s-1}(n2^{s-1})^{r-1}E_{n2^s} \le \sum_{k=n2^{s-1}+1}^{n2^s} k^{r-1}E_k, \quad s \ge 1.$$

Write down the series

$$f - t_n = \sum_{s=0}^{\infty}[t_{n2^{s+1}} - t_{n2^s}]$$

which converges uniformly in $\mathbb{T}$. The next chain of inequalities shows in particular that the series $\sum_{s=0}^{\infty}[t^{(r)}_{n2^{s+1}} - t^{(r)}_{n2^s}]$ converges uniformly in $\mathbb{T}$, whence $f^{(r)} - t_n^{(r)}$ exists and is equal to its sum. So,

$$\begin{aligned}
E_n(f^{(r)}) \le \|f^{(r)} - t_n^{(r)}\| &\le \textstyle\sum_{s=0}^{\infty} \|t^{(r)}_{n2^{s+1}} - t^{(r)}_{n2^s}\| \\
&\le (2n)^r\|t_{2n} - t_n\| + \textstyle\sum_{s=1}^{\infty}(n2^{s+1})^r\|t_{n2^{s+1}} - t_{n2^s}\| \\
&\le (2n)^r E_n(f) + 2^{2r+1}\textstyle\sum_{s=1}^{\infty}(n2^{s-1})^r E_{n2^s}(f) \\
&\le c_r[n^r E_n(f) + \textstyle\sum_{k=n+1}^{\infty} k^{r-1}E_k(f)].
\end{aligned}$$

Theorem 10.4 (The second inverse theorem) *We have*

$$\omega(f^{(r)}, \tfrac{1}{n}) \le c_r\Big[n^{-1}\sum_{k=0}^{n} k^r E_k(f) + \sum_{k=n+1}^{\infty} k^{r-1}E_k(f)\Big]\,.$$

Proof. Combine the estimates (10.2) and (10.3). □

10.3 Exercises

10.1. Prove the Bernstein inequality for algebraic polynomials

$$|p_n'(x)| \le \frac{n}{\sqrt{1-x^2}}\|p_n\|_\infty \quad \forall p_n \in \mathcal{P}_n[-1,1].$$

Hence derive the Markov inequality

$$\|p_n'\|_\infty \le n^2\,\|p_n\|_\infty \quad \forall p_n \in \mathcal{P}_n[-1,1].$$

Remark. Historically, Bernstein proved his inequalities the other way round: he proved first (rather non-trivially) the algebraic case and then derived the trigonometric version. Actually his arguments repeated those of A. Markov.

10.2. Complete the proof of Theorem 10.4.

10.3.* (*Direct theorems*) Show that the max-norm of the Fourier operator admits the estimate

$$\|s_n\|_\infty = \int_{-\pi}^{\pi} |D_n(t)|\,dt \le c \ln n.$$

Hence deduce that if the function $f \in C(\mathbb{T})$ satisfies the condition

$$\omega(f,t) = o(1/\ln \tfrac{1}{t}),$$

then $s_n(f)$ converge uniformly to f.