Part III - Lent Term 2005
Approximation Theory – Lecture 11

11 Degree of trigonometric approximation (cont.)

11.1 Approximation of functions from Lipschitz spaces

Definition 11.1 The Lipschitz space $\mathrm{Lip}\,\alpha$, $0 < \alpha \le 1$, consists of all functions $f \in C(\mathbb{T})$ which satisfy

$$|f(x) - f(y)| \le M|x-y|^\alpha,$$

or, equivalently, $\omega(f,t) \le Mt^\alpha$.

Theorem 11.2 *Let $f \in C(\mathbb{T})$ and $0 < \alpha < 1$. Then $E_n(f) = \mathcal{O}(n^{-\alpha})$ if and only if $f \in \mathrm{Lip}\,\alpha$.*

Proof. If $\omega(f,t) \le Mt^\alpha$, then by the first Jackson theorem $\quad E_n(f) \le c\omega(f,\frac{1}{n}) \le c'n^{-\alpha}$.

If $E_k(f) \le c\,k^{-\alpha}$, then by the first inverse theorem $\quad \omega(f,\frac{1}{n}) \le \frac{c}{n}\sum_{k=0}^n k^{-\alpha} \le cn^{-\alpha}$. □

Theorem 11.3 *Let $0<\alpha<1$. Then $E_n(f) = \mathcal{O}(n^{-r-\alpha})$ if and only if $f \in C^r(\mathbb{T})$ and $f^{(r)} \in \mathrm{Lip}\,\alpha$.*

Proof. The second Jackson plus the second inverse theorems. □

11.2 Zygmund classes

For $\alpha = 1$ the direct and the inverse estimates do not match. Namely,

$$\begin{array}{llll} (a) & \omega(f,\delta) = \mathcal{O}(\delta) & \Rightarrow & E_n(f) = \mathcal{O}(\frac{1}{n}), \\ (b) & E_n(f) = \mathcal{O}(\frac{1}{n}) & \Rightarrow & \omega(f,\delta) = \mathcal{O}(\delta \ln \frac{1}{\delta}). \end{array}$$

Moreover, in (b), it is really the estimate for $\omega(f,\delta)$ that is "bad", not the method of its proof, i.e., there exist functions f with $\omega(f,\delta) = \mathcal{O}(\delta \ln \frac{1}{\delta})$ and $E_n(f) = \mathcal{O}(\frac{1}{n})$ (see Exercise 11.1). It took more than 30 years before A. Zygmund found appropriate characterization of the functions with the approximation order $\mathcal{O}(\frac{1}{n})$.

Theorem 11.4 (Zygmund[1][1945]) *Let $f \in C(\mathbb{T})$. Then $E_n(f) = \mathcal{O}(\frac{1}{n})$ if and only if $\omega_2(f,\delta) = \mathcal{O}(\delta)$.*

Proof. The Jackson operator (see Exercise 9.3) provides the estimate

$$E_n(f) \le c\omega_2(f,\tfrac{1}{n}),$$

so that if $\omega_2(f,t) \le Mt$, then $E_n(f) \le \frac{1}{n}$.

On the other hand, Theorem 10.2 is easily generalized for the r-th modulus of smoothness:

$$\omega_r(f,\tfrac{1}{n}) \le c_r n^{-r} \sum_{k=0}^n k^{r-1} E_k(f). \tag{11.1}$$

For $r = 2$, from $E_k(f) \le \frac{c}{k}$ we obtain $\omega_2(f,\frac{1}{n}) \le \frac{c}{n^2}\sum_{k=0}^n 1 \le \frac{1}{n}$ as required. □

[1] Antoni Zygmund (1900-1992), Polish mathematician, since 1940 worked in USA, his "Trigonometric Series" is a classic that is still the definitive work on the subject.

11.3 Approximation by algebraic polynomials

Set $I = [-1, 1]$. If $f \in C(I)$, then $\widetilde{f}(\theta) := f(\cos\theta)$ is a 2π-periodic *even* continuous function and

$$\|f\|_{C(I)} = \|\widetilde{f}\|_{C(\mathbb{T})}. \tag{11.2}$$

Moreover, 1) if $p_n \in \mathcal{P}_n$, then $\widetilde{p}_n(\theta) = \sum_{k=0}^n a_k \cos k\theta$

2) if $t_n(\theta) = \sum_{k=0}^n a_k \cos k\theta$, then there exists an alg. pol. $p_n \in \mathcal{P}_n$ such that $t_n = \widetilde{p}$.

Lemma 11.5 *Let $f \in C(I)$ and $g = \widetilde{f}$. Then*

$$\omega(g, t) \le \omega(f, t). \tag{11.3}$$

Proof. We require the elementary inequality

$$|\cos\theta - \cos\xi| \le |\theta - \xi|.$$

For any $t > 0$, it provides inclusion of the sets $\{(\theta, \xi) : |\theta - \xi| < t\} \subset \{(\theta, \xi) : |\cos\theta - \cos\xi| < t\}$. Then

$$\begin{aligned} \omega(g, t) &:= \sup_{|\theta-\xi|\le t} |g(\theta) - g(\xi)| = \sup_{|\theta-\xi|\le t} |f(\cos\theta) - f(\cos\xi)| \\ &\le \sup_{|\cos\theta-\cos\xi|\le t} |f(\cos\theta) - f(\cos\xi)| = \sup_{|x-y|\le t} |f(x) - f(y)| \quad \le \quad \omega(f, t)\,. \end{aligned}$$

Theorem 11.6 (The first Jackson theorem) *If $f \in C(I)$, then*

$$E_n(f) \le c\,\omega(f, \tfrac{1}{n})\,.$$

Proof. Let $t_n \in \mathcal{T}_n$ be the best trigonometric approximation to $g(\theta) := \widetilde{f}(\theta) := f(\cos\theta)$. Then t_n is an *even* trigonometric polynomial, and with $p_n \in \mathcal{P}_n$ such that $t_n(\theta) = p(\cos\theta)$, making use of (11.2)-(11.3), we obtain

$$E_n(f) \le \|f - p_n\|_{C(I)} = \|\widetilde{f} - \widetilde{p}_n\|_{C(T)} =: \|g - t_n\| = \widetilde{E}_n(g) \le c\omega(g, \tfrac{1}{n}) \le c\omega(f, \tfrac{1}{n})\,.$$

Theorem 11.7 (The second Jackson theorem) *If $f \in C^{(r)}(I)$ and $n \ge r - 1$, then*

$$E_n(f) \le c_r\, n^{-r} \omega(f^{(r)}, \tfrac{1}{n})\,. \tag{11.4}$$

Proof. Exercise.

11.4 Exercises

11.1. Verify that, for $f(x) = x \ln \frac{\pi}{|x|}$, where $x \in [-\pi, \pi]$, and for δ small enough, we have

$$\omega(f, \delta) = \delta \ln \tfrac{1}{\delta}, \qquad \omega_2(f, \delta) = |\Delta_\delta^2 f(0)| = M\delta.$$

11.2. Prove the inverse estimate (11.1).

11.3. Prove the estimate (11.4). Show in between that

$$E_n(f) \le c n^{-1} E_{n-1}(f').$$

11.4. Prove that for $f(x) = \sqrt{1 - x^2}$, $x \in [-1, 1]$, and its algebraic approximation we have

$$\omega(f, \delta) = \mathcal{O}(\delta^{1/2}), \quad \text{but} \quad E_n(f) = \mathcal{O}(\tfrac{1}{n})$$

This shows in particular that the inverse theorems in the form they were given for the trigonometric approximation are *not* valid for the algebraic case.

11.5. For the Weierstrass function

$$f(x) := \sum_{k=1}^{\infty} \tfrac{1}{2^k} \cos 3^k x\,,$$

find the order of its modulus of continuity $\omega(f, t)$.