Part III - Lent Term 2005
Approximation Theory – Lecture 12

12 B-splines

12.1 Splines

Definition 12.1 (Splines) Let $N, k \in \mathbb{N}$, and let Δ_* be a knot sequence

$$\Delta_* = (a = \tau_0 < \tau_1 \cdots < \tau_N < \tau_{N+1} = b)\,.$$

The spline space $\mathcal{S}_k(\Delta_*)$ of order k (and deficiency 1) is the space of piecewise polynomial functions of degree $\le k-1$ on Δ_* which satisfy $k-1$ continuity conditions at each interior knot τ_i, i.e.,

$$s \in \mathcal{S}_k(\Delta_*) \quad \Leftrightarrow \quad \begin{array}{ll} 1) & s \in \mathcal{P}_{k-1}[\tau_i, \tau_{i+1}], \\ 2) & s \in C^{k-2}(\tau_{i-1}, \tau_{i+1}). \end{array}$$

If $s^{(k-1)}$ is continuous at τ_i, then s is a polynomial in a neighbourhood of τ_i, in this case the breakpoint τ_i is inessential.

Example 12.2 So, the splines of order 1 are step functions, those of order 2 are broken lines, and so on. A typical spline of order k is the truncated power

$$(x-\tau_i)_+^{k-1} = \begin{cases} (x-\tau_i)^{(k-1}, & x \ge \tau_i; \\ 0, & x < \tau_i. \end{cases}$$

Definition 12.3 (Basis) A basis for a finite-dimensional space $\mathcal{U}$ is a sequence of elements $(f_i)_{i=1}^n$ of $\mathcal{U}$ such that each $f \in \mathcal{U}$ has a unique representation $f = \sum_{i=1}^n a_i f_i$. The number n is the dimension of $\mathcal{U}$.

Lemma 12.4 *Suppose that there are elements $(f_i)_{i=1}^n$ of $\mathcal{U}$ and linear functionals $(a_i)_{i=1}^n$ on $\mathcal{U}$ such that*

$$1) \quad a_i(f_j) = \delta_{ij}, \qquad 2) \quad a_i(f) = 0 \quad \textit{all } i \quad \Rightarrow \quad f = 0.$$

Then (f_i) is a basis for $\mathcal{U}$.

Proof. Given f, set $g := \sum_{i=1}^n a_i(f) f_i$. Then, by (1), $a_j(g) = a_j(f)$ all j, so that, by (2), $g = f$, i.e.,

$$f = \sum_{i=1}^n a_i(f) f_i\,.$$

If $\sum_{i=1}^n b_i f_i$ is another representation, then applying a_j to both ones, we obtain $b_j = a_j(f)$. □

Theorem 12.5 *The space $\mathcal{S}_k(\Delta_*)$ has the basis*

$$\begin{aligned} s_{0j}(x) &:= \tfrac{1}{(k-j)!}(x-a)^{k-j}, \quad j = 1..k; \\ s_i(x) &:= \tfrac{1}{(k-1)!}(x-\tau_i)_+^{k-1}, \quad i = 1..N; \end{aligned}$$

with the dual functionals

$$\begin{aligned} a_{0j}(s) &:= s^{(k-j)}(a), && j = 1..k; \\ a_i(s) &:= s^{(k-1)}(\tau_i+) - s^{(k-1)}(\tau_i-), && i = 1..N. \end{aligned}$$

In particular,

$$\dim \mathcal{S}_k(\Delta_*) = k + N =: n\,.$$

Proof. The duality $a_i(s_j) = \delta_{ij}$ is straightforward. Let $s \in \mathcal{S}_k(\Delta_*)$. If $a_i(s) = 0$ for $i = 1..N$, then $s^{(k-1)}$ is continuous at each τ_i, hence (as $s^{(k-1)}$ is piecewise constant) $s^{(k-1)}$ is a constant, therefore $s \in \mathcal{P}_{k-1}[a,b]$. If also $a_{0j}(s) = 0$, then all the derivatives of s are zeros, hence $s \equiv 0$. □

Corollary 12.6 *Each $s \in \mathcal{S}_k(\Delta_*)$ admits a representation*

$$s(x) = p_{k-1}(x) + \textstyle\sum_{i=1}^N c_i (x-\tau_i)_+^{k-1}\,. \tag{12.1}$$

Remark 12.7 The basis of truncated powers is quite inconvenient for numerical computations: the elements have large support, the basis itself is unstable (it becomes almost linear dependent when $\tau_i \to \tau_{i+1}$). Curry and Schoenberg discovery of the basis of B-splines made a revolution.

12.2 Divided differences

Definition 12.8 (Divided difference) Given $f \in C[a,b]$ and a sequence of $(k+1)$ points $(t_0,\dots,t_k)$, the divided difference $f[t_0...t_k]$ of order k is the leading coefficient of the Lagrange polynomial $p \in \mathcal{P}_k$ which interpolates f at these points (i.e., the coefficient at x^k in $p(x)$). By definition,

$$f[t_0...t_k] = 0 \quad \text{if} \quad f \in \mathcal{P}_{k-1}, \quad \text{and} \quad [t_0...t_k]\,x^k = 1.$$

Remark 12.9 (Multiple points) If the sequence (t_i) has multiple entries, i.e., if

$$(t_0, t_1, \dots, t_k) := \big(\underbrace{\tau_1,\dots,\tau_1}_{m_1}, \dots, \underbrace{\tau_\ell,\dots,\tau_\ell}_{m_\ell}\big),$$

then $p \in \mathcal{P}_k$ is the Lagrange–Hermite interpolating polynomial to f:

$$p^{(s-1)}(\tau_i) = f^{(s-1)}(\tau_i), \quad s = 1...m_i, \quad i = 1...\ell, \quad \sum m_i = k+1.$$

Properties 12.10 Let us recall some properties of the divided differences.

1) *Explicit formula.* If all the points (t_i) are distinct, then

$$f[t_0...t_k] = \sum_{\nu=0}^k \frac{f(t_\nu)}{\omega'(t_\nu)}, \quad \omega(x) = \prod_{\nu=0}^k (x - t_\nu).$$

This follows from the representation of the Lagrange polynomial $p(x) = \sum_{\nu=0}^k \frac{f(t_\nu)}{\omega'(t_\nu)} \frac{\omega(x)}{x-t_\nu}$ by identifying its leading coefficient.

2) *Recurrence relation.* The following formula allows to compute the divided difference of any order k adaptively, starting with the values $f[t_i] = f(t_i)$:

$$f[t_0...t_k] = \frac{f[t_1...t_k] - f[t_0...t_{k-1}]}{t_k - t_0}, \quad \forall\, t_0 \ne t_k. \tag{12.2}$$

This follows from the formula $p(x) = \frac{x-t_0}{t_k-t_0} p_0(x) + \frac{t_k - x}{t_k - t_0} p_k(x)$, which relates the Lagrange polynomial p of degree k that interpolates f on $t = (t_i)_{i=0}^k$ with two Lagrange polynomials p_j of degree $k-1$ that interpolate f on the sets $t \setminus t_j$, respectively.

3) *Convexity.* The previous formula implies that, for $t = (t_0, ..., t_k)$, we also have

$$f[t \setminus t_j] = \gamma f[t \setminus t_k] + (1-\gamma) f[t \setminus t_0], \quad t_j = \gamma t_k + (1-\gamma) t_0. \tag{12.3}$$

4) *Leibniz rule.* If $h = fg$, then

$$h[t_0...t_k] = \sum_{i=0}^k f[t_0...t_i]\, g[t_i...t_k]\,. \tag{12.4}$$

12.3 Exercises

12.1. Let $(\ell_i)_{i=0}^n$ be the sequence of Lagrange fundamental polynomials with respect to a sequence of points $(t_i)_{i=0}^n$. Find a set of functionals (a_i) dual to (ℓ_i).

12.2. Prove that each $s \in \mathcal{S}_k(\Delta_*)$ can be also written in the form

$$s(x) = q_{k-1}(x) + \sum_{i=1}^{N} b_i(\tau_i - x)_+^{k-1}\,.$$

(We use $(\tau_i - x)_+^r$ instead of $(x - \tau_i)_+^r$ in (12.1).) What is the relation between b_i and c_i?

12.3. Consider the space $\mathcal{S} = \mathcal{S}_2(\Delta_*)$ of linear splines on the knot sequence

$$(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4) = (0, 1, 2, 3, 4)$$

Find the representaion of the form (12.1) for $s_i \in \mathcal{S}$, where s_i is given by

$$s_i(\tau_j) = \delta_{ij}, \quad j = 0..4, \quad i = 2, 3.$$

12.4. Prove that the Lagrange–Hermite interpolating polynomial exists and is unique.

12.5. Using Rolle's theorem prove that $f[t_0...t_j] = \frac{1}{j!}f^{(j)}(\xi)$ for some $\xi \in [t_0, t_j]$. Hence, from formula (12.4), deduce the Leibnitz rule for derivatives: *If $f, g \in C^k[a, b]$, and $h = fg$, then*

$$h^{(k)}(x) := [f(x)g(x)]^{(k)} = \sum_{i=0}^{k} \binom{k}{i} f^{(i)}(x)g^{(k-i)}(x).$$

12.6. Prove the Leibnitz rule by induction. For $k = 0$, clearly, $h[t_0] = f[t_0]g[t_0]$. Then

$$(t_k - t_0)h[t_0...t_k] = h[t_1...t_k] - h[t_0...t_{k-1}] = \cdots$$