Part III - Lent Term 2005
Approximation Theory – Lecture 13

13 B-splines (cont.)

13.1 Basic properties of B-splines

Definition 13.1 (Curry, Schoenberg[1][1966]) Given $k, n \in \mathbb{N}$, and a knot sequence

$$\Delta = \{a \le t_1 \le t_2 \le \ldots \le t_{n+k} = b\}, \quad t_i < t_{i+k},$$

the B-spline sequences $(M_i)_{i=1}^n$ and $(N_i)_{i=1}^n$ of order k are defined as

$$\begin{array}{rcll} M_i(t) & := & k\,[t_i \ldots t_{i+k}](\cdot - t)_+^{k-1} & (\text{Normalization } \int M_i(t)\,dt = 1), \\ N_i(t) & := & (t_{i+k} - t_i)\,[t_i \ldots t_{i+k}](\cdot - t)_+^{k-1} & (\text{Normalization } \sum_i N_i \equiv 1), \\ M_i(t) & = & \frac{k}{t_{i+k} - t_i}\, N_i(t) & (\text{Relation}). \end{array} \tag{13.1}$$

Properties 13.2 B-splines have the following properties.

1) *Spline structure.* B-spline M_i is indeed a spline, i.e., a piecewise polynomial function of order k with the knots $t_i, \ldots, t_{i+k}$. In particular, when this knots are distinct, we have the explicit representation

$$M_i(t) = k \sum_{\nu=i}^{i+k} \frac{(t_\nu - t)_+^{k-1}}{\omega'(t_\nu)}\,, \tag{13.2}$$

which shows that $M_i \in \mathcal{S}_k(\Delta) \subset C^{k-2}[a, b]$. (If a knot t_j appears m times in the sequence $(t_i \ldots t_{i+k})$, then $M_i \in C^{k-1-m}$ in a neighbourhood of t_j.) This follows from the formula for divided diffrences.

2) *Finite support*. B-splines have a finite support

$$\operatorname{supp} M_i = [t_i, t_{i+k}]\,,$$

because if $t \ge t_{i+k}$, then $(x - t)_+^{k-1} = 0$ at $x = t_i, \ldots, t_{i+k}$, hence $[t_i \ldots t_{i+k}](x - t)_+^{k-1} = 0$;
and if $t \le t_i$, then $(x - t)_+^{k-1} = (x - t)^{k-1}$ at $x = t_i, \ldots, t_{i+k}$, hence $[t_i \ldots t_{i+k}](x - t)_+^{k-1} = 0$.

3) *Peano kernel for divided difference*. If $f \in C^k[a, b]$, then taking the Taylor formula

$$f(x) = p_{k-1}(x) + \frac{1}{(k-1)!} \int_a^b (x - t)_+^{k-1} f^{(k)}(t)\,dt$$

and applying divided differences to both sides, we obtain

$$f[t_i \ldots t_{i+k}] = \frac{1}{k!} \int_a^b M_i(t) f^{(k)}(t)\,dt, \tag{13.3}$$

i.e., B-spline M_j is the Peano kernel in the integral representation of the functional $[t_i \ldots t_{i+k}]$.

4) *B-splines normalization.* For any Δ, the B-splines obey the following normalization conditions

$$a) \int_a^b M_i(t)\,dt = 1 \qquad b) \sum_i N_i(t) \equiv 1, \quad t \in [t_k, t_{n+1}] \tag{13.4}$$

[1] Isaaac Schoenberg (1903-1990), the father of the spline theory, it was his paper of 1946 where the word spline appeared, in 1930 married E. Landau's daughter Charlotte in Berlin, this was not his only mathematical connection by marriage since his sister married Hans Rademacher.

4a) The first condition follows from (13.3) if we take $f(x) = x^k$.

4b) For the second one, we use definition of N_i and properties of divided differrences. So,

$$N_i(t) := (t_{i+k} - t_i)\,[t_i...t_{i+k}](\cdot - t)_+^{k-1} = ([t_{i+1}...t_{i+k}] - [t_i...t_{i+k-1}])\,(\cdot - t)_+^{k-1},$$

thus

$$\sum_{i=1}^{n} N_i(t) = \big([t_{n+1}...t_{n+k}] - [t_1...t_k]\big)(x-t)_+^{k-1}.$$

If $t \ge t_k$, then $(x-t)_+^{k-1} = 0$ at $x = t_1, ..., t_k$, hence $[t_1...t_k](x-t)_+^{k-1} = 0.$

If $t \le t_{n+1}$, then $(x-t)_+^{k-1} = (x-t)^{k-1}$ at $x = t_{n+1}, ..., t_{n+k}$, hence $[t_{n+1}...t_{n+k}](x-t)_+^{k-1} = 1,$

5) *Recurrence relation* (de Boor[2][1972]). The following formula relates two adjacent B-splines of order $k-1$ with the supports $[t_i, t_{i+k-1}]$ and $[t_{i+1}, t_{i+k}]$ with that of oder k with the overlapping support $[t_i, t_{i+k}]$

$$\frac{1}{k} M_{i,k}(t) = \frac{1}{k-1}\left[\frac{t - t_i}{t_{i+k} - t_i} M_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_i} M_{i+1,k-1}(t)\right]. \tag{13.5}$$

For the proof, we notice that

$$(x-t)_+^{k-1} = (x-t)\cdot(x-t)_+^{k-2},$$

so it make sense to find a relation between the following Lagrange polynomials:

$$\begin{array}{lll} p_0 \in \mathcal{P}_{k-1} & \text{that interpolates } f(x) & \text{on } (t_0, t_1, \dots t_{k-1}), \\ p_1 \in \mathcal{P}_{k-1} & \text{that interpolates } f(x) & \text{on } \quad (t_1, \dots t_{k-1}, t_k), \\ p \in \mathcal{P}_k & \text{that interpolates } (x-t)f(x) & \text{on } (t_0, t_1, \dots t_{k-1}, t_k). \end{array}$$

Since

$$x - t = \gamma_t (x - t_k) + (1-\gamma_t)(x - t_0), \quad \gamma_t = \frac{t - t_0}{t_k - t_0}, \quad 1 - \gamma_t = \frac{t_k - t}{t_k - t_0},$$

it follows that

$$p(x) = \gamma_t (x - t_k) p_0(x) + (1-\gamma_t)(x - t_0) p_1(x).$$

Hence, for the divided differences (as they are the leading coefficients) we obtain

$$[t_0...t_k](\cdot - t) f = \gamma_t [t_0...t_{k-1}] f + (1-\gamma_t)[t_1...t_k] f\,,$$

so substituting $f(x) = (x-t)_+^{k-2}$ and using the definition of the B-splines M we derive (13.5).

From (13.5), using the relation (13.1) between two types of B-splines, one obtains the recurrence formula for N_i which is used more often and have a bit different form

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t). \tag{13.6}$$

Notice that that the values at the denominators are the lengths of the corresponding supports.

5) *Positivity.* Since $N_{i,1}$ is just a step function with support on $[t_i, t_{i+1}]$, namely

$$N_{i,1} := \chi_{[t_i, t_{i+1}]} = \begin{cases} 1, & x \in [t_i, t_{i+1}); \\ 0, & \text{otherwise}; \end{cases} \tag{13.7}$$

the recurrence (13.6) provides immediately finiteness of support of B-splines and their positivity

$$\operatorname{supp} N_{i,k} = [t_i, t_{i+k}], \qquad N_{i,k} > 0 \quad \text{on} \quad (t_i, t_{i+k}),$$

as well as a piecewise polynomial structure. Notice, however, that the continuity conditions at knots do not follow immediately and require additional proof. One can start with (13.6)-(13.7) as a definition of B-splines, and then derive all further properties.

[2]Carl de Boor, b.1937, another father of spline theory, his web-site at www.cs.wisc.edu/˜ deboor/deboor.html is worth of visiting, actually the spline part of this course follows his notes.

13.2 Exercises

13.1. We will write $N_i(x) = N(x; t_i, ...t_{i+k})$ to make the dependence on knots explicit.

Using the recurrence relation (13.6), prove that, for the so-called Bernstein knots

$$\Delta = \{t_{-k} = \cdots = t_{-1} = 0 < 1 = t_1 = \cdots = t_k\},$$

the corresponding B-splines are

$$N_{i,k}(x) := N(x; \underbrace{0...0}_{k+1-i}, \underbrace{1...1}_{i}) = \binom{k-1}{i-1} x^{i-1}(1-x)^{k-i}, \quad i = 1, ..., k,$$

i.e., $(N_{i,k})$ are the Bernstein basis polynomials (used in the proof of Korovkin theorem). You may assume without the proof that, for all k,

$$N_{1,k}(x) = (1-x)^{k-1}, \quad N_{k,k}(x) = x^k.$$

13.2 Let $t_{k-i} = \cos \frac{\pi i}{k}$, $i = 0...k$, i.e., t_{k-i} are the points of equioscillation of the Chebyshev polynomial T_k. Prove that the spline $M(x) := M(x; t_0, ..., t_k)$ is the so-called *perfect* spline, i.e.,

$$|M^{(k-1)}(x)| = \text{const}$$

Hint. From the formula (15.2) find the value $c_\nu := M_i^{(k-1)}(x)$, $x \in [t_\nu, t_{\nu+1})$. You may need the formula

$$(x^2-1)T_n''(x) + xT_n'(x) = n^2 T_n(x).$$