## Part III - Lent Term 2005
# Approximation Theory – Lecture 14

## 14 B-splines (cont.)

### 14.1 B-splines as basis functions

**Definition 14.1** Given $\Delta = (t_i)_{i=1}^{n+k}$, let $\omega_i$, $\psi_i$ and $\ell_i(\cdot, t)$ be polynomials in $\mathcal{P}_{k-1}$ defined as

1) $\omega_i(x) := (x - t_{i+1}) \cdots (x - t_{i+k-1})$,
2) $\psi_i(x) := \frac{1}{(k-1)!}\omega_i(x)$ (to avoid factorials),
3) $\ell_i(\cdot, t)$ interpolates $(\cdot - t)_+^{k-1}$ on $x = t_i, ..., t_{i+k-1}$.

**Lemma 14.2 (Lee's formula)** *For any $\Delta = (t_i)_{i=1}^{n+k}$, we have*

$$\omega_i(x) N_i(t) = \ell_{i+1}(x, t) - \ell_i(x, t), \quad \forall x, t \in \mathbb{R}\,. \tag{14.1}$$

**Proof.** For a fixed $t$, the difference on the right-hand side of (14.1), as a function of $x$, is a polynomial of degree $k-1$ that is equal to zero at $x = t_{i+1}, ..., t_{i+k-1}$, hence

$$\ell_{j+1}(\cdot, t) - \ell_j(\cdot, t) = c(t)\omega_i(\cdot)\,.$$

The value of the constant $c(t)$ is the leading coefficient of this polynomial, i.e., the difference of leading coefficients of polynomials $\ell_{i+1}(\cdot, t)$ and $\ell_i(\cdot, t)$. These are, however, the Lagrange interpolants to $(\cdot - t)_+^{k-1}$, therefore their leading coefficients are just corresponding divided differences, hence

$$c(t) = ([t_{i+1} ... t_{i+k}] - [t_i ... t_{i+k-1}])(\cdot - t)_+^{k-1} =: N_i(t).$$

**Proposition 14.3 (Marsden's identity)** *For any $k, n \in \mathbb{N}$, and for any knot sequence $\Delta = (t_i)_{i=1}^{n+k}$,*

$$(x - t)^{k-1} = \sum_{i=1}^{n} \omega_i(x) N_i(t), \quad t_k \le t \le t_{n+1}, \quad \forall x \in \mathbb{R}. \tag{14.2}$$

**Proof.** Summing both sides of Lee's formula (14.1), we obtain

$$\sum_{i=1}^{n} \omega_i(x) N_i(t) = \sum_{i=1}^{n} \ell_{i+1}(x, t) - \ell_i(x, t) = \ell_{n+1}(x, t) - \ell_1(x, t).$$

Further arguments are similar to those used for proving $\sum_i N_i \equiv 1$ in (13.4). We have

$$\begin{aligned} \ell_1(\cdot, t) &\quad \text{interpolates } (\cdot - t)_+^{k-1} \text{ at } x = t_1, ..., t_k \\ \ell_{n+1}(\cdot, t) &\quad \text{interpolates } (\cdot - t)_+^{k-1} \text{ at } x = t_{n+1}, ..., t_{n+k}; \end{aligned}$$

so that

if $t \ge t_k$, then $\ell_1(x) = (x - t)_+^{k-1} = 0$ at $x = t_1, ..., t_k$, hence $\ell_1(x, t) \equiv 0$;
if $t \le t_{n+1}$, then $\ell_{n+1}(x) = (x - t)_+^{k-1} = (x - t)^{k-1}$ at $x = t_{n+1}, ..., t_{n+k}$, hence $\ell_{n+1}(x, t) \equiv (x - t)^{k-1}$.

This proves the statement. $\square$

**Corollary 14.4** *The polynomials $\mathcal{P}_{k-1}[t_k, t_{n+1}]$ belong to* $\operatorname{span}(N_i)_{i=1}^n$

**Proof.** With $\psi_i := \frac{1}{(k-1)!}\omega_i$, the Marsden identity (14.2) takes the form $\frac{(x-t)^{k-1}}{(k-1)!} = \sum_{i=1}^n \psi_i(x)N_i(t)$. On differentiating this relation $m-1$ times with respect to $x$, and interchanging $x$ and $t$ on the left-hand side, we obtain

$$\frac{1}{(k-m)!}(t-x)^{k-m} = \sum_{i=1}^n (-1)^{k-m}\psi_i^{(m-1)}(x)N_i(t), \quad m = 1, ..., k. \tag{14.3}$$

In particular, $\{(t-a)^{k-m}\}_{m=1}^k$, hence all polynomials of degree $k-1$, belong to $\mathrm{span}(N_i)_{i=1}^n$ on $[t_k, t_{n+1}]$.

**Example 14.5** Here are two special cases of (14.3) of particular interest.

1) For $m = k$, as $\psi_i^{(k-1} \equiv 1$, we are getting the already familiar equality

$$1 \equiv \sum_{i=1}^n N_i(t), \quad t \in [t_k, t_{n+1}],$$

which reminds once again that the B-splines $(N_i)$ form a *partition of unity*,

2) For $m = k-1$, we obtain the following relation

$$t - x = \sum_{i=1}^n (-1)\left[x - \frac{t_{i+1} + \cdots + t_{i+k-1}}{k-1}\right] N_i(t),$$

which implies that for all *linear* polynomials $p(t) = a_1 t + a_0$, with the notation $t_i^* := \frac{t_{i+1} + \cdots + t_{i+k-1}}{k-1}$, we have the equality

$$p(t) = \sum_{i=1}^n p(t_i^*)N_i(t), \qquad \forall p \in \mathcal{P}_1. \tag{14.4}$$

**Corollary 14.6** *If $t_j$ are distinct, then the truncated power $(t-t_j)_+^{k-1}$ belongs to $\mathrm{span}(N_i)_{i=1}^n$*

**Proof.** In (14.3), take $x = t_j$ and $m = 1$. If $t_i < t_j < t_{i+k}$, then $\psi_i(t_j) = 0$, hence

$$\frac{1}{(k-1)!}(t-t_j)^{k-1} = \Big(\sum_{i+k\le j} + \sum_{i\ge j}\Big)(-1)^{k-1}\psi_i(t_j)N_i(t).$$

For $t \ge t_j$, since the B-splines $(N_i)_{i+k\le j}$ have supports to the left of $t_j$, the first sum vanishes. On the other hand, the remaining second sum vanishes for $t \le t_j$. Hence,

$$\frac{1}{(k-1)!}(t-t_j)_+^{k-1} = \sum_{i\ge j}(-1)^{k-1}\psi_i(t_j)N_i(t).$$

**Theorem 14.7** *The B-spline sequence $(N_i)_{i=1}^n$ on $\Delta$ forms a basis for the space $\mathcal{S}_k(\Delta_*)$ where*

$$\begin{array}{llccccccl}
\Delta_*: & & a=\tau_0 & < & \tau_1 & <\cdots< & \tau_N & < & \tau_{N+1}=b \\
\Delta: & t_1 \le \cdots \le t_{k-1} \le & a=t_k & < & t_{k+1} & <\cdots< & t_n & < & t_{n+1}=b \quad \le t_{n+1} \le \cdots \le t_{n+k}
\end{array}$$

**Proof.** We have just proved that each element of the basis of truncated powers

$$\begin{aligned} s_{0j}(x) &:= \tfrac{1}{(k-j)!}(x-a)^{k-j}, \quad j = 1...k\,; \\ s_i(x) &:= \tfrac{1}{(k-1)!}(x-\tau_i)_+^{k-j}, \quad i = 1...N; \end{aligned}$$

(which is a basis by Theorem 12.5) is spanned by $(N_i)_{i=1}^n$. Since the number of functions in both sequences is equal (to $n = N+k$), the B-spline sequence is a basis, too. $\square$

## 14.2 Exercises

**14.1.** More about B-splines. From the definition (13.1), derive that

$$N'_{i,k}(t) = M_{i,k-1}(t) - M_{i+1,k-1}(t)$$

whence, using the recurrence relation (13.6), show that

$$N_{i,k}^{(m)}(t) = \frac{k-m}{k}\left[\frac{t-t_i}{t_{i+k-1}-t_0}N_{i,k-1}^{(m)}(t) + \frac{t_{i+k}-t}{t_{i+k}-t_1}N_{i+1,k-1}^{(m)}(t)\right].$$

**14.2.** Let $(t_i)$ be the integer knot sequence, i.e., $t_i = i$. Prove that

$$N_i^{(k-1)}(t) = \text{const}\,(-1)^m \binom{k-1}{m}, \quad t \in (t_{i+m}, t_{i+m+1})$$

Of course, you need to take only one particular $i$, say $i = 0$, since the corresponding B-spline sequence $(N_i)$ is *shift invariant*, i.e., $N_i(t) = N_{i-1}(t-1)$.

**14.3.** Denote by $t_i^{(r)}$ the elementary symmetric f-ns of $k-1$ variables $t_{i+1}, .., t_{i+k-1}$ of degree $r$,

$$\begin{aligned}
t_i^{(0)} &= 1, \\
t_i^{(1)} &= t_{i+1} + \cdots + t_{i+k-1}, \\
t_i^{(2)} &= t_{i+1}t_{i+2} + t_{i+1}t_{i+3} + \cdots t_{i+k-2}t_{i+k-1}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
t_i^{(k-1)} &= t_{i+1}t_{i+2}\cdots t_{i+k-1}.
\end{aligned}$$

Derive from the Marsden identity that monomials $(\cdot)^r$ have the representation

$$t^r = \textstyle\sum_{i=1}^n \binom{k-1}{r}^{-1} t_i^{(r)} N_i(t).$$