Part III - Lent Term 2005
Approximation Theory – Lecture 15

15 Dual functionals

15.1 Construction

In this section, given a knot-sequence $(t_i)_{i=1}^{n+k}$ and the B-splines $(N_i)_{i=1}^{n}$, we construct a sequence of dual functionals (μ_i) such that

$$\mu_i(N_j) = \delta_{ij}\,.$$

We will seek such functionals in the form

$$\mu_i(f) = \int_a^b f(t)\phi_i(t)\,dt,$$

and moreover we will require later from a function ϕ_i to have support in $[t_i, t_{i+k}]$, so that the range of integration in the right-hand side will be actually $\int_{t_i}^{t_{i+k}}$. To proceed further, we recall that the B-spline $M_j = cN_j$ is the Peano kernel in the integral representation of the divided difference $[t_j...t_{j+k}]$, i.e., for any $h \in C^k[a,b]$,

$$\int_a^b M_j(t)h^{(k)}(t) = k!\,h[t_j...t_{j+k}]\,.$$

So, it make sense to consider ϕ_i in the form $\phi_i = h_i^{(k)}$, and then look for a particular h_i whose divided differences vanish for all sets $[t_j...t_{j+k}]$ but one.

Lemma 15.1 *Given $\Delta = (t_i)$, let h_i be a function such that $h_i \in C^k[a,b]$ and*

$$h_i(t_j) = 0 \quad t_j \le t_{i+k-1}; \qquad h_i(t_j) = \psi_i(t_j) \quad t_j \ge t_{i+1}\,. \tag{15.1}$$

Then the functional $\mu_i : C[a,b] \to \mathbb{R}$ given by the rule

$$\mu_i(f) := \int_a^b \phi_i(t)f(t)\,dt, \quad \phi_i = h_i^{(k)},$$

satisfies $\mu_i(N_j) = \delta_{ij}$.

Proof. We have

$$\mu_i(N_j) = \int_a^b N_j(t)h_i^{(k)}(t)\,dt = c_j\,\frac{1}{k!}\int_a^b M_j(t)h_i^{(k)}(t)\,dt = c_j\,h_i[t_j...t_{j+k}],$$

where $c_j = (t_{j+k} - t_j)(k-1)!$. But, due to definition (15.1), all the divided differences $h_i[t_j...t_{j+k}]$ vanish except that for $j = i$. Indeed, if $t_j \le t_{i-1}$, or $t_j \ge t_{i+1}$, then the values $h_i(t_j), ..., h_i(t_{j+k})$ coincide with the values of an algebraic polynomial p of degree $k-1$, namely $p \equiv 0$ or $p = \psi_i$, respectively, hence the corresponding divided difference of order k is zero. For $j = i$ we have

$$(t_{i+k} - t_i)\,h_i[t_i...t_{i+k}] = h_i[t_{i+1}...t_{i+k}] - h_i[t_i...t_{i+k-1}] = \psi_i[t_{i+1}...t_{i+k}] = 1/(k-1)!$$

whence $c_i h_i[t_i...t_{i+k}] = 1$. □

Remark 15.2 The simplest way to construct a function h_i satisfying (15.1) is to set

$$h_i(x) \equiv 0, \quad x \le \eta_i; \qquad h_i(x) = \psi_i(x), \quad x \ge \xi_i, \qquad \eta_i, \xi_i \in [t_i, t_{i+k}], \tag{15.2}$$

and then to blend smoothly the values in between by posing the conditions

$$h_i^{(m-1)}(\eta_i) = 0, \quad h_i^{(m-1)}(\xi_i) = \psi_i^{(m-1)}(\xi_i), \quad m = 1, ..., k, \tag{15.3}$$

(and adding the conditions $h_i(x) = 0$ at $x = t_{i+1}, ..., t_{i+k-1}$, when necessary). This construction will also provide

$$\mathrm{supp}\, \phi_i = \mathrm{supp}\, h_i^{(k)} = [\eta_i, \xi_i] \subset [t_i, t_{i+k}]$$

because by (15.2) the function h_i coincide with a polynomial of degree $k-1$ on either side of $[t_i, t_{i+k}]$.

Lemma 15.3 *Let $p \in \mathcal{P}_{k-1}$. Then the functional μ_i defined by* (15.3) *satisfies*

$$\mu_i(p) = \textstyle\sum_{m=1}^{k} (-1)^{k-m} \psi_i^{(m-1)}(\xi) p^{(k-m)}(\xi) =: \lambda_i(p, \xi), \tag{15.4}$$

Proof. Integrating by parts gives

$$\mu_i(f) = \int_\eta^\xi p(t) h^{(k)}(t)\, dt = \Big[p(t) h^{(k-1)}(t) - p'(t) h^{(k-2)}(t) + \cdots + (-1)^{k-1} p^{(k-1)}(t) h(t) \Big]_{t=\eta}^{t=\xi},$$

where the integral remainder vanishes because $p^{(k)} \equiv 0$. It remains to notice that, by definition, at ξ all derivatives of h coincide with those of ψ, while at η they are equal to zero. □

Remark 15.4 The functionals $\lambda_i(f, x)$ defined by (15.4) are called the de Boor-Fix functionals.

15.2 Boundedness of the functionals μ_i

Take $[\eta_i, \xi_i] = [t_i, t_{i+k}]$. The simplest way to construct a function h_i satisfying (15.2)-(15.3) is to set

$$h_i(x) := g_i(x) \psi_i(x),$$

where $g_i \in C^k[a, b]$ is obtained from any fixed function g by the formula

$$g_i(x) = g(\tfrac{x - t_i}{t_{i+k} - t_i}), \quad g(x) = \begin{cases} 0, & k\text{-fold at } x = 0; \\ 1, & k\text{-fold at } x = 1. \end{cases} \tag{15.5}$$

(Fulfillment of (15.3) can be verified by using the Leibnitz rule).

Theorem 15.5 *The functionals*

$$\mu_i(f) = \int_{t_i}^{t_{i+k}} \phi_i(t) f(t)\, dt, \quad \phi_i := [g_i \psi_i]^{(k)},$$

form a dual basis to the B-spline basis (N_i), and they admit the estimate

$$|\mu_i(f)| \le c_k |I_i|^{-1/p} \|f\|_{L_p(I_i)}, \quad I_i = [t_i, t_{i+k}].$$

In particular, they are bounded linear functionals on $C(I_i)$.

Proof. From the Leibnitz rule, we find

$$|\phi_i(x)| := [g_i(x) \psi_i(x)]^{(k)} \le 2^k \max_{0 \le m \le k} |g_i^{(m)}(x)|\, |\psi_i^{(k-m)}(x)|\,.$$

The first factor, as a derivative of a composite function, admits the estimate

$$|g_i^{(m)}(x)| = |g^{(m)}(x)| |t_{i+k} - t_i|^{-m} \le c_k\, |t_{i+k} - t_i|^{-m},$$

where the constant c_k is a bound for the first k derivatives of the function g. The second factor is a polynomial of degree $m-1$ with leading coefficient $a_{m-1} = \frac{1}{(m-1)!}$ and with all its zeros, say x_j, lying inside $[t_i, t_{i+k}]$. Hence,

$$|\psi_i^{(k-m)}(x)| = a_{m-1} |(x - x_1) \cdots (x - x_{m-1})| \le |t_{i+k} - t_i|^{m-1}.$$

Therefore $\|\phi_i\|_{L_\infty(I_i)} \le c_k\, |I_i|^{-1}$, and

$$\|\phi_i\|_{L_q(I_i)} \le |I_i|^{1/q} \|\phi_i\|_{L_\infty(I_i)} \le c_k |I_i|^{1/q - 1} = c_k |I_i|^{-1/p}, \quad 1/p + 1/q = 1.$$

By Hölder inequality $|\int \phi f| \le \|\phi\|_q \|f\|_p$, whence the statement. □

15.3 Exercises

15.1. For some functions f, g and for some $x \in \mathbb{R}$, define the functional

$$\lambda(f, g, x) := \sum_{m=1}^{k} (-1)^{k-m} g^{(m-1)}(x) f^{(k-m)}(x),$$

whenever the right-hand side exists. If we take $g(x) = \psi_i(x)$, then we obtain the de Boor–Fix functional $\lambda_i(f, x) := \lambda(\psi_i, f, x)$ (see formula 15.4).

Prove that, for any two polynomials $p, q \in \mathcal{P}_{k-1}$, the functional $\lambda(p, q, x)$ is independent of x, i.e.,

$$\lambda(p, q, x) \equiv \mathrm{const}(p, q).$$

15.2. Prove that the formula (15.4) remains valid for splines in $\mathcal{S}_k(\Delta)$, i.e., that the functional μ_i defined by (15.3) also satisfies

$$\mu_i(s) = \textstyle\sum_{m=1}^{k} (-1)^{k-m} \psi_i^{(m-1)}(\xi) s^{(k-m)}(\xi) =: \lambda_i(s, \xi), \quad \forall\, \xi \in [t_i, t_{i+k}]\,.$$

15.3. Derive formula (15.4) from identities (14.3).

Hint. Put identities (14.3) in the Taylor expansion of $p(t)$ at x

$$p(t) = \sum_{j=0}^{k-1} \frac{1}{j!} (t - x)^j p^{(j)}(x) = \sum_{m=1}^{k} p^{(k-m)}(x) \frac{(t - x)^{k-m}}{(k - m)!} = \cdots =$$