Part III - Lent Term 2005
Approximation Theory – Lecture 16

16 Dual functionals (cont.)

16.1 Quasi-interpolants and degree of spline approximation

We may reformulate Theorem 15.5 in the following way.

Theorem 16.1 *Any spline $s \in \mathcal{S}_k(\Delta)$ has the B-spline expansion*

$$s(t) = \textstyle\sum_{i=1}^n \mu_i(s) N_i(t), \tag{16.1}$$

where $\mu_i : C[t_i, t_{i+k}] \to \mathbb{R}$ are locally supported dual functionals (15.6) *whose norms are uniformly bounded by some constant c_k independently of Δ.*

Definition 16.2 (Quasi–interpolant) Given $k, n \in \mathbb{N}$ and Δ, the quasi-interpolant is a mapping $Q : C[a,b] \to \mathcal{S}_k(\Delta)$ given by the formula

$$Q(f,t) := \textstyle\sum_{i=1}^n \mu_i(f) N_i(t). \tag{16.2}$$

By (16.1), if $s = \sum a_i N_i$, then $\mu_i(s) = a_i$, hence $\mu_i(Q(f)) = \mu_i(f)$ for $i = 1, ..., n$. The name quasi-interpolant was chosen to emphasize that $Q(f)$ does not actually interpolate f at some nodal points, but matches another information about f (that is enough to reproduce the polynomials).

Lemma 16.3 *The mapping Q is a projector from $C[a,b]$ to $\mathcal{S}_k(\Delta)$. It satisfies the local estimate*

$$\|Q(f)\|_{C[t_j,t_{j+1}]} \le c_k \|f\|_{C[t_{j+1-k},t_{j+k}]}.$$

Proof. It is clear (from (16.1)-(16.2), say) that Q is a projector. For the estimate we employ finiteness of support of B-splines, that they form a partition of unity, and uniform boundedness of dual functionals. For $x \in [t_j, t_{j+1}]$, we obtain

$$\begin{aligned} |Q(f,x)| &:= \Big|\sum_{i=1}^n \mu_i(f) N_i(x)\Big| = \Big|\sum_{i=j+1-k}^{j} \mu_i(f) N_i(x)\Big| \\ &\le \max_{j+1-k \le i \le j} |\mu_i(f)| \sum_{i=j+1-k}^{j} |N_i(x)| \le c_k \max_{j+1-k \le i \le j} \|f\|_{C[t_i,t_{i+k}]} = c_k \|f\|_{C[t_{j+1-k},t_{j+k}]}. \end{aligned}$$

Theorem 16.4 *The quasi-interpolant Q admits the estimate*

$$\|f - Q(f)\|_{C[t_j,t_{j+1}]} \le c_k \operatorname{dist}(f, \mathcal{P}_{k-1})_{C[t_{j+1-k},t_{j+k}]} \le c_k' \, |t_{j+k} - t_{j+1-k}|^k \|f^{(k)}\|_{C[t_{j+1-k},t_{j+k}]}$$

Proof. Since $Q(f)$ is a projector on the space of polynomials with the bounded norm, we may locally apply the Lebesgue inequality. Precisely, $f - Q(f) = [I - Q](f - p)$ for any $p \in \mathcal{P}_{k-1}$, hence

$$\|f - Qf\|_{C[t_j,t_{j+1}]} \le (1 + \|Q\|_{C[t_j,t_{j+1}]}) \operatorname{dist}(f, \mathcal{P}_{k-1})_{C[t_{j+1-k},t_{j+k}]} \, .$$

The second estimate is obtained by scaling the Jackson estimate to arbitrary interval (or by Lagrange interpolation, say). □

Theorem 16.4 readily provides a direct estimate for $E_{k,\Delta}(f)$, the value of best approximation of f by splines of order k with a knot sequence Δ.

Theorem 16.5 (Direct estimate) *For any k, Δ we have*

$$E_{k,\Delta}(f) \le c_k |t|^k \|f^{(k)}\|_\infty, \quad |t| := \max_i |t_{i+1} - t_i| \tag{16.3}$$

Remark 16.6 (Inverse estimate) However, having $E_\Delta(f) = \mathcal{O}(|t|^k)$ is, in general, no guarantee that $f \in W_\infty^k$ unless the knot sequences Δ involved are sufficiently generic. Indeed, if every knot sequence contains the point $\frac{1}{2}$, then the function $f(x) = (x - \frac{1}{2})_+^{k-1}$ can be approximated without error from $\mathcal{S}_k(\Delta)$ even though f fails to have a kth derivative. What is true, however, is that the estimate $E_{k,\Delta}(f) = \mathcal{O}(|t|^k)$ cannot hold for *every* knot sequence Δ unless $f \in W_\infty^k$. This conclusion can be already reached if $E_{k,\Delta}(f) = \mathcal{O}(|t|^k)$ holds for every *uniform* knot sequence, namely *for spline approximation $E_{k,n}(f)$ of order k on the uniform knot sequences with n knots, we have*

$$E_{k,n}(f) = \mathcal{O}(n^{-k}) \quad \Leftrightarrow \quad f \in W_\infty^k,$$
$$E_{k,n}(f) = o(n^{-k}) \quad \Leftrightarrow \quad f \in \mathcal{P}_{k-1}.$$

The second relation shows the so-called *saturation order*. Thus, no function different from a polynomial of degree $k-1$ can be approximated by splines of degree $k-1$ on the uniform meshes with the order better than $\mathcal{O}(n^{-k})$. The non-empty class of functions that enjoys this order, W_∞^k in this case, is called the *saturation class*.

16.2 B-spline basis condition number

Definition 16.7 Let $\Phi = (\phi_i)$ be a basis for a subspace $\mathcal{U}$ in $L_p[a,b]$. The number

$$\kappa_p := \sup_a \frac{\|a\|_{\ell_p}}{\|\sum_i a_i u_i\|_{L_p}} \cdot \sup_b \frac{\|\sum_i b_i u_i\|_{L_p}}{\|b\|_{\ell_p}}$$

is called the p-condition number of Φ. It measures the extent to which the relative changes in the coordinates of an element of $\mathcal{U}$ may be close to the resulting relative change in an element itself. For the spline space $\mathcal{S}_k(\Delta)$ we define its L_p-normalized basis $(\widehat{N}_i)$ as

$$\widehat{N}_i := (\tfrac{k}{|I_i|})^{1/p} N_i = M_i^{1/p} N_i^{1-1/p}.$$

Theorem 16.8 *The B-spline basis p-condition number κ_p is bounded undependently of Δ. Precisely, there exists a number d_k such that*

$$d_k^{-1}\, \|a\|_{\ell_p} \le \|\textstyle\sum_i a_i \widehat{N}_i\|_{L_p} \le \|a\|_{\ell_p}. \tag{16.4}$$

Proof. (*Upper estimate.*) We use the known relations $\sum_i |N_i| \equiv 1$ and $\int_a^b |M_i| = 1$, and Hölder inequality: $\sum |a_i b_i| \le \|a\|_{\ell_p} \|b\|_{\ell_q}$ where $1/p + 1/q = 1$. So,

$$\begin{aligned} \|\textstyle\sum a_i \widehat{N}_i\|_{L_p} &= \|\textstyle\sum a_i M_i^{1/p} N_i^{1/q}\|_{L_p} \le \|\,(\sum |a_i|^p M_i)^{1/p} \cdot (\sum N_i)^{1/q}\,\|_{L_p} \\ &\le \|\,(\textstyle\sum |a_i|^p M_i)^{1/p}\,\|_{L_p} = \|\,\sum |a_i|^p M_i\,\|_{L_1}^{1/p} = (\,\sum |a_i|^p\,)^{1/p} =: \|a\|_{\ell_p} \end{aligned}$$

(*Lower estimate.*) Let $s = \sum a_i \widehat{N}_i$. To get an ℓ_p-bound for the coefficients (a_i), we rewrite this p-normalized B-spline expansion in terms of the standard (N_i), i.e., $s = \sum a_i (\frac{k}{|I_i|})^{1/p} N_i$, and apply then the bound for the functionals (μ_i) dual to (N_i):

$$|a_i| (\tfrac{k}{|I_i|})^{1/p} = |\mu_i(s)| \le d_k (\tfrac{1}{|I_i|})^{1/p} \|s\|_{L_p(I_i)}$$

This results in in the inequalities $|a_i|^p \le d_k^p k^{-1} \int_{I_i} |s|^p$ and summing them up gives

$$\|a\|_{\ell_p}^p = \sum |a_i|^p \le d_k^p k^{-1} \sum \int_{I_i} |s|^p = d_k^p \int_a^b |s|^p = d_k^p \|s\|_{L_p[a,b]}^p\,.$$

(The last but one equality is because every interval $[t_j, t_{j+1}]$ is covered by k intervals $I_i = [t_i, t_{i+k}]$.)

Remark 16.9 A basis ϕ that satisfies the inequalities

$$c_2\, \|a\|_{\ell_2} \le \|\sum_i a_i \phi_i\|_{L_2} \le c_1\, \|a\|_{\ell_2}$$

is called a Riesz basis.

16.3 Exercises

16.1. Prove that the quasi-interpolation by *linear* splines

$$Q(f,x) = \sum_{i=1}^{n} f(t_{i+1}) N_i(x)$$

(which is now a true interpolation) satisfies the estimate

$$\|f - Q(f)\| \le \tfrac{1}{8} |t|^2 \, \|f''\| \, .$$

Hint. Do *not* use Theorem 16.4.

16.2. The Schoenberg spline operator $S : C[a,b] \to \mathcal{S}_k(\Delta)$ is given by the formula

$$S(f,t) := S_\Delta(f,t) := \sum_i f(t_i^*) N_i(t), \qquad t_i^* = \tfrac{t_{i+1} + \cdots + t_{i+k-1}}{k-1} \tag{1}$$

[compare with the formula (14.4)]. Prove that

$$\|f - S(f)\|_{C[t_j, t_{j+1}]} \le 2 \mathrm{dist}(f, \mathcal{P}_1)_{C[t_{j+1-k}, t_{j+k}]} \, ,$$

hence derive that

$$\|f - S(f)\| \le c_k \, |t|^2 \|f''\| \, ,$$

and try to minimize the constant c_k as much as possible.

16.3.* Prove that the quasi-interpolant Q is a bounded operator in L_p-norm, namely that

$$\|Q(f)\|_{L_p[a,b]} \le d_k \|f\|_{L_p[a,b]} \, .$$

Hint. Write $Q(f)$ in the form

$$Q(f,t) := \sum_{i=1}^{n} \widehat{\mu}_i(f) \widehat{N}_i(t),$$

where $(\widehat{N}_i)$ is the L_p-normalized B-spline basis, and use the estimates from the proof of Theorem 16.8