Part III - Lent Term 2004

Approximation Theory – Lecture 18

18 B-splines (cont.)

18.1 B-Splines with multiple knots

Lemma 18.1 *For the knots* $(t_i, ..., t_{i+k}) = (\underbrace{\tau_r, .., \tau_r}_{m_r}, ..., \underbrace{\tau_s, .., \tau_s}_{m_s})$, *the divided difference has the form*

$$f[t_i...t_{i+k}] = \sum_{\mu=r}^{s} \sum_{\nu=1}^{m_\mu} a_{\mu\nu} f^{(\nu-1)}(\tau_\mu). \tag{18.1}$$

Proof. Let $p(x) = \sum_{j=0}^{k} b_j x^j$ be the Hermite polynomial interpolant to f on $(t_i, ..., t_{i+k})$, i.e.,

$$p^{(\nu-1)}(\tau_\mu) = f^{(\nu-1)}(\tau_\mu), \quad \nu = 1...m_\mu, \quad \mu = r...s.$$

The coefficients of p, in particular, the leading one, are linear functions of the values interpolated.

Lemma 18.2 *Let* $(t_1, ..., t_{n+k}) = (\tau_0, \underbrace{\tau_1, .., \tau_1}_{k-1}, ..., \underbrace{\tau_N, .., \tau_N}_{k-1}, \tau_{N+1})$. *Then,*

$$\begin{aligned} c\,N(t; \underbrace{\tau_j..\tau_j}_{k+1-m}, \underbrace{\tau_{j+1}..\tau_{j+1}}_{m}) &= (t-\tau_j)^{m-1}(\tau_{j+1}-t)^{k-m}, \quad 1 < m < k; \\ c\,N(t; \tau_{j-1}, \underbrace{\tau_j..\tau_j}_{k-1}, \tau_{j+1}) &= \begin{cases} (t-\tau_{j-1})^{k-1}, & t \in [\tau_{j-1}, \tau_j]\,; \\ (\tau_{j+1}-t)^{k-1}, & t \in [\tau_j, \tau_{j+1}]\,. \end{cases} \end{aligned}$$

Proof. By (18.1),

$$N(t; \underbrace{\tau_j..\tau_j}_{k+1-m}, \underbrace{\tau_{j+1}..\tau_{j+1}}_{m}) = \sum_{\nu=1}^{k+1-m} a_\nu (\tau_j - t)_+^{k-\nu} + \sum_{\nu=1}^{m} b_\nu (\tau_{j+1} - t)_+^{k-\nu},$$

hence N belongs to $C^{m-2}(\tau_j)$ and to $C^{k-m-1}(\tau_{j+1})$. Since also $N \equiv 0$ outside (τ_j, τ_{j+1}), it follows that N has an $(m-1)$-multiple zero at τ_j and a $(k-m)$-multiple zero at τ_{j+1}. But N is also a polynomial of degree $k-1$ on $[\tau_j, \tau_{j+1}]$, hence the result. ∎

Corollary 18.3 *For* $(t_1, ..., t_{n+k}) = (\tau_0, \underbrace{\tau_1, .., \tau_1}_{k-1}, ..., \underbrace{\tau_N, .., \tau_N}_{k-1}, \tau_{N+1})$, *we have*

$$N_j(t_i) = \begin{cases} 1, & \textit{if} \quad t_j < t_i < t_{j+k}; \\ 0, & \textit{otherwise.} \end{cases}$$

18.2 Knot insertion

Let $\widehat{\Delta}$ be obtained from Δ by just one *knot insertion*, i.e., $\widehat{t}_i = t_i$ for $i < j$ and $\widehat{t}_i = t_{i-1}$ for $i > j$,

$$\begin{aligned} \Delta &= t_1 \le \cdots \le t_{j-1} & & \le t_j & \le \cdots \le t_{n+k}, \\ \widehat{\Delta} &= \widehat{t}_1 \le \cdots \le \widehat{t}_{j-1} & \le \widehat{t}_j & \le \widehat{t}_{j+1} & \le \cdots \le \widehat{t}_{n+1+k}. \end{aligned}$$

Then the spline space $\mathcal{S}_k(\Delta)$ on the coarse grid is a subspace of $\mathcal{S}_k(\widehat{\Delta})$ on a finer grid, and we are interested in connection between coefficients of expansions of an element $s \in \mathcal{S}_k(\Delta)$ in two different bases $s := \sum a_i N_i = \sum \widehat{a}_i \widehat{N}_i$.

Lemma 18.4 *For any (a_i), we have*

$$s := \sum_{i=1}^{n} a_i N_i = \sum_{i=1}^{n+1} \widehat{a}_i \widehat{N}_i, \quad \textit{where} \quad \widehat{a}_i = (1-\gamma_i)\, a_{i-1} + \gamma_i a_i, \tag{18.2}$$

and $\gamma_i = \max\left(0, \min\left(1, \frac{\widehat{t}_j - \widehat{t}_i}{\widehat{t}_{i+k} - \widehat{t}_i}\right)\right)$.

Proof. The coefficients of the both expansions are determined via dual functionals as $a_i = \lambda_i(s)$ and $\widehat{a}_i = \widehat{\lambda}_i(s)$, and the functionals λ_i are defined by the values of $\omega_i(x)$. For $\widehat{t}_j \in [\widehat{t}_i, \widehat{t}_{i+k}] = [t_i, t_{i+k-1}]$, we have the equality

$$(x - \widehat{t}_j) = (1-\gamma_i)\,(x - \widehat{t}_i) + \gamma_i\,(x - \widehat{t}_{i+k}),$$

and it follows that

$$\widehat{\omega}_i(x) = (1-\gamma_i)\,\omega_{i-1}(x) + \gamma_i\,\omega_i(x),$$

hence the same equality for λ_is and therefore for a_is. ■

Remark 18.5 If we make the $(k-1)$-fold insertion of a knot $x = \widehat{t}_j$, then for the resulting B-spline basis $(\widetilde{N}_i)$ all but one values $(\widetilde{N}_i(x))$ are equal to zero, hence $s(x) = \sum \widetilde{a}_i \widetilde{N}_i(x) = \widetilde{a}_j$. This provides an algoritm of evaluation of a spline s from its B-spline coefficients.

18.3 Sign variation

Definition 18.6 For a real sequence $a = (a_i)$ without zero entries, denote by

$$S(a) := \#\{i : a_i a_{i+1} < 0\}$$

the number of *sign changes* in it. If a has zero entries, then the minimal number of sign changes obtainable in such a sequence by prescribing some signs to zero entries is denoted by

$$S^-(a) := \min_{\varepsilon} S(a^\varepsilon), \quad a^\varepsilon = (a_i + \varepsilon_i),$$

and is called the number of *strong sign changes*. (The maximal number so obtained is denoted by $S^+(a)$ and is called the number of *weak sign changes*.) If we set $a_0 :=$ (a with zero entries deleted), then $S^-(a) = S(a_0)$. If f is a function on an interval, one defines

$$S^-(f) := \sup_{x_1 < \cdots < x_r} S^-(f(x_1), ..., f(x_r)).$$

Example 18.7 $\quad a = (-1, 0, -1, 0, -1), \quad S^+(a) = 4, \quad S^-(a) = 0;$

$\quad a = (-1, 0, +1, 0, -1), \quad S^+(a) = 2, \quad S^-(a) = 2.$

Lemma 18.8 *If $s = \sum_{j=1}^{n} a_j N_j = \sum_{j=1}^{n+1} \widehat{a}_j \widehat{N}_j$, then*

$$S^-(\widehat{a}) \le S^-(a).$$

Proof. Draw the sequence (a_i) as the graph Γ of a broken line (t_i^*, a_i) with some increasing (t_i^*). Each $\widehat{a}_i$ is a convex combination of a_i and a_{i-1}, hence we can represent $(\widehat{a}_i)$ as the the graph $\widehat{\Gamma}$ of a broken line $(\widehat{t}_i^*, \widehat{a}_i)$ which vertices lie on Γ. The statement follows from the geometric picture. ■

Theorem 18.9 (Karlin [1968]) *For any $s = \sum a_j N_j$ from $\mathcal{S}_k(\Delta)$, we have $S^-(s) \le S^-(a)$.*

Proof. Let $x_1 < \cdots < x_r$ be any finite sequence of real numbers. Insert each of x_i and each of t_j enough times so that the resulting refined knot sequence $\widetilde{\Delta}$ contains each x_i and each t_j exactly $k-1$ times, and let $s = \sum \widetilde{a}_j \widetilde{N}_j$ be expansion with respect to the refined B-psline sequence.

By Corollary 18.3, there is exactly one B-spline, say N_j, which is nonzero at x_i, hence $\widetilde{a}_j = s(x_i)$. So, $(s(x_i))$ is a subsequence of $\widetilde{a}$, therefore,

$$S^-(s(x_1), ..., s(x_r)) \le S^-(\widetilde{a}) \le S^-(a),$$

the second inequality by the repeated use of previous lemma. Since (x_i) is arbitrary, the statement follows. ■

18.4 Exercises

18.1. Let $\omega(x) = (x - t_0)...(x - t_k)$, $x_i \neq x_j$. Prove that

$$\frac{1}{\omega(x)} = [t_0, ..., t_k]\,\frac{1}{(x - \cdot)}$$

18.2. Assuming the continuity of $f[t_0...t_k]$ with respect to t_j for sufficiently smooth function, prove that, for

$$(t_0, ..., t_k) = (\underbrace{\tau_r, .., \tau_r}_{m_r}, ..., \underbrace{\tau_s, .., \tau_s}_{m_s}),$$

the coefficients $a_{\mu\nu}$ in the formula (20.3) satisfy

$$\frac{1}{\omega(x)} = \sum_{\mu=r}^{s} \sum_{\nu=1}^{m_\mu} a_{\mu\nu} \frac{(\nu - 1)!}{(x - \tau_\mu)^\nu}.$$

In other words, they provide the decomposition of $\frac{1}{\omega(x)}$ into *partial fractions*.

18.3 Let $s = \sum_i a_i N_i$ and let $t_i^* := (t_{i+1} + ... + t_{i+k-1})/(k-1)$. Since $x = \sum_i t_i^* N_i(x)$, we can represent the spline *curve* $(x, s(x))$ as a vector-valued spline $\sum_i P_i N_i$ with coefficients $P_i = (t_i^*, a_i)$ called its *control points*. The broken line Γ_s with the vertices (t_i^*, a_i) is called the *control polygon*.

Verify that, by the knot insertion, the points t_i satisfy

$$\widehat{t_i^*} = \gamma_i t_i^* + (1 - \gamma_i) t_{i-1}^*,$$

i.e., the same relation as for the refined B-spline coefficients.

Therefore (compare with the proof of Lemma 18.7), the refined control polygon $\widehat{\Gamma}_s$ interpolates, at its breakpoints, the original one. If $\widehat{t}_j$ is inserted $(k-1)$ times, then $\widehat{\Gamma}_s(\widehat{t}_j) = s(\widehat{t}_j)$.