Part III - Lent Term 2004

Approximation Theory – Lecture 19

Theorem 18.9 (Karlin [1968]) *For any* $s = \sum a_j N_j$ *from* $\mathcal{S}_k(\Delta)$*, we have* $S^-(s) \le S^-(a)$.

Proof. Let $x_1 < \cdots < x_r$ be any finite sequence of real numbers. Insert each of x_i and each of t_j enough times so that the resulting refined knot sequence $\widetilde{\Delta}$ contains each x_i and each t_j exactly $k-1$ times, and let $s = \sum \widetilde{a}_j \widetilde{N}_j$ be expansion with respect to the refined B-spline sequence.

By Corollary 18.3, there is exactly one B-spline, say N_j, which is nonzero at x_i, hence $\widetilde{a}_j = s(x_i)$. So, $(s(x_i))$ is a subsequence of $\widetilde{a}$, therefore,

$$S^-(s(x_1), ..., s(x_r)) \le S^-(\widetilde{a}) \le S^-(a),$$

the second inequality by the repeated use of Lemma 18.8. Since (x_i) is arbitrary, the statement follows. ∎

19 Spline interpolation (cont.)

19.1 Spline interpolation projector

Definition 19.1 Any $\boldsymbol{x} = (x_1 < \cdots < x_n)$ satisfying the Schoenberg-Whitney condition for $\mathcal{S}_k(\Delta)$ gives rise to the (interpolating) projector

$$P_{\boldsymbol{x}} : C[a,b] \to \mathcal{S}_k(\Delta), \quad f \to P_{\boldsymbol{x}} f = s$$

that associates f with the unique spline $s \in \mathcal{S}_k(\Delta)$ that agrees with f on $\boldsymbol{x}$. Since $P_{\boldsymbol{x}}$ is a linear projector, by Lebesgue inequality,

$$E_{k,\Delta}(f) \le \|f - P_{\boldsymbol{x}} f\| \le (1 + \|P_{\boldsymbol{x}}\|) E_{k,\Delta}(f)$$

so that $P_{\boldsymbol{x}}$ with a small norm is a candidate for good approximation method.

Lemma 19.2 *Let* $A_{\boldsymbol{x}} = (N_j(x_i))$ *with* $\operatorname{diag} A_{\boldsymbol{x}} > 0$. *Then*

$$\|P_{\boldsymbol{x}}\|_{L_\infty} \le \|A_{\boldsymbol{x}}^{-1}\|_{\ell_\infty}. \tag{19.1}$$

Proof. We have

$$P_{\boldsymbol{x}} f = \sum a_j N_j \quad \text{where} \quad A_{\boldsymbol{x}} a = s\big|_{\boldsymbol{x}} = f\big|_{\boldsymbol{x}}, \quad \text{i.e.,} \quad a = A_{\boldsymbol{x}}^{-1}(f\big|_{\boldsymbol{x}}).$$

Since $\sum |N_j| = 1$, we derive

$$\|P_{\boldsymbol{x}} f\|_{L_\infty} = \|\sum a_j N_j\|_{L_\infty} \le \max |a_j| = \|a\|_{\ell_\infty},$$

hence

$$\|P_{\boldsymbol{x}} f\| \le \|a\|_{\ell_\infty} = \|A_{\boldsymbol{x}}^{-1}(f\big|_{\boldsymbol{x}})\|_{\ell_\infty} \le \|A_{\boldsymbol{x}}^{-1}\|_{\ell_\infty} \|f\big|_{\boldsymbol{x}}\|_{\ell_\infty} \le \|A_{\boldsymbol{x}}^{-1}\|_{\ell_\infty} \|f\|_{L_\infty}.$$

Therefore,

$$\|P_{\boldsymbol{x}}\| := \sup_f \frac{\|P_{\boldsymbol{x}} f\|_{L_\infty}}{\|f\|_{L_\infty}} \le \|A_{\boldsymbol{x}}^{-1}\|_{\ell_\infty}.$$ ∎

Remark 19.3 Hence, in search for a better approximation from $\mathcal{S}_k(\Delta)$, we now look for $\boldsymbol{x}$ that makes $\|A_{\boldsymbol{x}}^{-1}\|$ as small as possible. It must be not surprising that this search leads to the Chebyshev equioscillating splines. The main intermediate tool is *total positivity* of the spline collocation matrix $A_{\boldsymbol{x}}$ to be established next.

19.2 Total positivity of the collocation matrix

Definition 19.4 (Totally positive matrices) A matrix $A \in \mathbb{R}^{m\times n}$ (not necessarily square) is called *totally positive* if all its minors are non-negative, i.e., for any strictly increasing (index) sequences

$$\boldsymbol{i} = (1 \le)\, i_1 < \cdots < i_r\ (\le m), \quad \text{and} \quad \boldsymbol{j} = (1 \le)\, j_1 < \cdots < j_r\ (\le n),$$

the determinant of the matrix $A(\boldsymbol{i},\boldsymbol{j}) := (A(i_p, j_q))_{p,q=1}^r$ is nonnegative, $\det A(\boldsymbol{i},\boldsymbol{j}) \ge 0$.

Theorem 19.5 (Karlin [1967]) *For any $\boldsymbol{x}$, any k, and any Δ, the collocation matrix*

$$A_{\boldsymbol{x}} := (N_j(x_i))_{i,j=1}^n$$

is totally positive.

Proof. Let $\widehat{\Delta}$ be obtained from Δ by the insertion of just one knot, and let $(\widehat{N}_j)$ be the B-spline sequence on $\widehat{\Delta}$. Then, by Lemma 18.4,

$$N_j = \alpha_j \widehat{N}_j + \alpha'_{j+1}\widehat{N}_{j+1}, \quad \alpha \ge 0.$$

Since the determinant of a matrix is a linear function of the columns of that matrix, we have

$$\det[..., N_j(x), ...] = \alpha_j \det[..., \widehat{N}_j(x), ...] + \alpha'_{j+1}\det[..., \widehat{N}_{j+1}(x), ...]$$

with elements $\cdots$ unchanged in their respective places. Since only neighbouring columns of $\widehat{A}$ participate in a column of A, it follows that, for any $\boldsymbol{i},\boldsymbol{j}$, we have the expansion

$$\det A(\boldsymbol{i},\boldsymbol{j}) = \sum_{\boldsymbol{h}} \gamma_{\boldsymbol{h}} \det \widehat{A}(\boldsymbol{i},\boldsymbol{h}), \quad \gamma_{\boldsymbol{h}} \ge 0, \tag{19.2}$$

with the sum over certain nondecreasing sequences $\boldsymbol{h}$. However, if an $\boldsymbol{h}$ is not strictly increasing, then the corresponding determinant is trivially zero, hence we may assume that, in (19.2), all sequences $\boldsymbol{h}$ are strictly increasing.

Now insert each of x_i and each of t_j enough times so that the resulting refined knot sequence $\widetilde{\Delta}$ contains each x_i and each t_j exactly $k-1$ times. By induction we have

$$\det A(\boldsymbol{i},\boldsymbol{j}) = \sum_{\boldsymbol{h}} \gamma_{\boldsymbol{h}} \det \widetilde{A}(\boldsymbol{i},\boldsymbol{h})$$

with $\gamma_{\boldsymbol{h}} \ge 0$ and all $\boldsymbol{h}$ strictly increasing. But now, in each row of $\widetilde{A}$ there is exactly one nonzero entry, namely the one belonging to $\widetilde{N}_j$ with

$$\widetilde{t}_j < x_i = \widetilde{t}_{j+1} = \cdots = \widetilde{t}_{j+k-1} < \widetilde{t}_{j+k},$$

and that entry equals 1. Further, since x_i are increasing, the only non-zero in the next row occurs to the right of this one. Thus $\widetilde{A}(\boldsymbol{i},\boldsymbol{h})$ has a nonzero entry in every row iff it is the identity matrix. Consequently, $\det \widetilde{A}(\boldsymbol{i},\boldsymbol{h})$ equals either 1 or 0, and therefore $\det A(\boldsymbol{i},\boldsymbol{j}) \ge 0$. ∎

Lemma 19.6 *Let A be invertible and totally positive. Then its inverse is a checkerboard matrix meaning*

$$(-1)^{i+j}A^{-1}(i,j) \ge 0,$$

and if a is a vector such that $Aa = (-1,1,-1,1,...)^T$, then

$$\|A^{-1}\|_\infty = \|a\|_\infty.$$

Proof. By Cramer's rule,

$$A^{-1}(i,j) = (-1)^{i+j}\frac{\det A(\backslash j, \backslash i)}{\det A}$$

while both determinants are nonnegative due to the total positivity. Now, the checkerboard structure implies

$$\|A^{-1}\|_\infty = \max_i \sum_j |A^{-1}(i,j)| = \max_i \sum_j A^{-1}(i,j)(-1)^{i+j} = \max_i (-1)^i a_i =: \|a\|_\infty$$

∎

19.3 Exercises

19.1. (*Exam question 2001*)

1) Consider the following case of parabolic spline (actually, polynomial) interpolation:

$$I = [0, 1], \quad k = 3, \quad \boldsymbol{x} = (0, \tfrac{1}{2}, 1), \quad \Delta = (0, 0, 0, 1, 1, 1).$$

The B-splines on the Bernstein knots are the fundamental Bernstein polynomials

$$N_1(x) = (1 - x)^2, \quad N_2(x) = 2x(1 - x), \quad N_3(x) = x^2.$$

Prove that $\|A_{\boldsymbol{x}}^{-1}\|$ provides the bound $\|P_{\boldsymbol{x}}\| \le 3$.

2) Prove directly that the exact value of the norm (of parabolic interpolation on $\boldsymbol{x}$) is $\|P_{\boldsymbol{x}}\| = \frac{5}{4}$, hence the bound (21.1) is not sharp.

19.2. Prove that the matrix

$$A = \begin{bmatrix} 1 & \cdots & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

is totally positive.

19.3. Let A be invertible and totally positive. If a, b are the vectors such that $Aa = b$ and $(-1)^i b_i \ge \gamma > 0$, then

$$\|A^{-1}\|_\infty \le \tfrac{1}{\gamma} \, \|a\|_\infty.$$