Part III - Lent Term 2004

Approximation Theory – Lecture 20

20 Spline interpolation (cont.)

20.1 Optimal interpolation

Let $A_{\boldsymbol{x}} := (N_j(x_i))$ be the B-spline collocation matrix, and let $a_{\boldsymbol{x}}$ be the unique vector such that $A_{\boldsymbol{x}} a^{\boldsymbol{x}} = (-1)^i$. Then

$$\|A_{\boldsymbol{x}}^{-1}\|_\infty = \|a^{\boldsymbol{x}}\|_\infty,$$

so that our search for $\min \|A_{\boldsymbol{x}}^{-1}\|_\infty$ is the search for $\boldsymbol{x}$ that minimizes $\|a^{\boldsymbol{x}}\|_\infty$.

Algorithm 20.1 (Remez algorithm) Let

$$s_{\boldsymbol{x}} := \sum_j a_j^{\boldsymbol{x}} N_j, \quad \text{so that} \quad s_{\boldsymbol{x}}(x_i) = (-1)^i, \quad i = 1...n.$$

We note that

$$(-1)^i a_i^{\boldsymbol{x}} > 0 \quad \forall i,$$

since $a_i^{\boldsymbol{x}} = \sum_j A_{\boldsymbol{x}}^{-1}(i,j)(-1)^j = (-1)^i \sum_j |A_{\boldsymbol{x}}^{-1}(i,j)|$ with the sum necessarily positive.

Since $s_{\boldsymbol{x}}$ strictly alternates in sign at the n points $(t_1 <)\ x_1 < \cdots < x_n\ (< t_{n+k})$ and vanishes at t_1 and at t_{n+k}, it follows that $s_{\boldsymbol{x}}$ has n distinct local extrema $y_1 < \cdots < y_n$ with

$$(-1)^i s_{\boldsymbol{x}}(y_i) \ge 1 \quad \forall i. \tag{20.1}$$

This implies that

$$t_i < y_i < t_{i+k} \quad \forall i.$$

Indeed, if, e.g., $y_i \le t_i$ for some i, then $N_j(y_1) = \cdots = N_j(y_i) = 0$ for all $j \ge i$. Hence, on $y_1, ..., y_i$, $s_{\boldsymbol{x}} := \sum_{j=1}^n a_j^{\boldsymbol{x}} N_j$ agrees with $\sum_{j=1}^{i-1} a_j^{\boldsymbol{x}} N_j$, i.e., the latter sum has at least $i-1$ strong sign changes. Then, by Karlin theorem 18.9, $i - 2 = S^-(a_1^{\boldsymbol{x}}, ..., a_{i-1}^{\boldsymbol{x}}) \ge S^-(\sum_{j=1}^{i-1} a_j^{\boldsymbol{x}} N_j) \ge i - 1$, a contradiction.

Consequently, since Schoenberg-Whitney conditions for $\boldsymbol{y}$ and Δ are satisfied, there is exactly one $s_{\boldsymbol{y}}$ with

$$s_{\boldsymbol{y}} =: \sum_j a_j^{\boldsymbol{y}} N_j, \qquad s_{\boldsymbol{y}}(y_i) = (-1)^i \quad \forall i. \tag{20.2}$$

Now, by (20.1) and (20.2), for any $\gamma < 1$, the difference $s_{\boldsymbol{x}} - \gamma s_{\boldsymbol{y}}$ strictly alternates in sign on $y_1 < \cdots < y_n$, hence (by Karlin theorem 18.9) we must have $S^-(a^{\boldsymbol{x}} - \gamma a^{\boldsymbol{y}}) = n - 1$, and therefore

$$(-1)^i a_i^{\boldsymbol{x}} \ge (-1)^i a_i^{\boldsymbol{y}} \ge 0 \quad \forall i.$$

In this way, we obtain a sequence $s_m := \sum_j a_j^m N_j$, $m = 1, 2, \ldots$, whose coefficients converge monotonely to some sequence a^*, and if we let $s_* := \sum_j a_j^* N_j$, then $s_m \to s_*$ uniformly in $[t_1, t_{n+k}]$. The corresponding sequences $y_1^m < \cdots < y_n^m$ of extrema of s_m therefore also converge, necessarily to a strictly increasing sequence $x_1^* < \cdots < x_n^*$ since s_m strictly alternates in sign on $y_1^m < \cdots < y_n^m$. Therefore,

$$1 = (-1)^i s_*(x_i^*) = \|s_*\|_\infty, \quad i = 1, \ldots, n, \tag{20.3}$$

since $1 = (-1)^i s_m(y_i^{m-1})$, while $\|s_m\| = \max_i |s_m(y_i^m)|$ and s_* is the uniform limit of (s_m).

Remark 20.2 The above algorithm does not guarantee that, starting with different $\boldsymbol{x}$, we will not arrive to different $\boldsymbol{x}^*$. It is the next lemma that shows that the limiting sequence is one and same.

Lemma 20.3 *Let $s_* = \sum_j a_j^* N_j$ satisfy* (20.3) *for some increasing $\boldsymbol{x}^*$. Then*

$$(-1)^i a_i^* = \|\lambda_i\|,$$

where (λ_i) are the functionals dual to (N_j).

Proof. Let $A_* := (N_j(x_i^*))$ and define the functional λ_i by the rule

$$\lambda_i : \mathcal{S}_k(\Delta) \to \mathbb{R} : s \to \sum_j A_*^{-1}(i,j) s(x_j^*).$$

Then, by definition ofA_*,

$$\lambda_i(N_s) = \delta_{is}.$$

i.e., λ_i are functionals dual to the B-spline basis. Also, $|\lambda_i(s)| \le \sum_j |A_*^{-1}(i,j)| \cdot \|s\|_\infty$, and the equality is attained for $s = s_*$ because

$$|\lambda_i(s_*)| = |\sum_j A_*^{-1}(i,j)(-1)^j| = \sum_j |A_*^{-1}(i,j)| = \sum_j |A_*^{-1}(i,j)| \, \|s_*\| \, .$$

This shows that $\|\lambda_i\| = \sum_j |A_*^{-1}(i,j)|$, and that $|\lambda_i(s_*)| = \|\lambda_i\|$. But we have also $|\lambda_i(s_*)| = |a_i^*|$ and $(-1)^i a_i^* > 0$, hence the result. ∎

Definition 20.4 (Chebyshev spline) The function s_* is, by definition, the Chebyshev spline for $\mathcal{S}_k(\Delta)$. It is the unique element that maximally equioscillates, say, between ± 1.

Theorem 20.5 (Demko [1982]) *For any knot sequence Δ there exists an interpolating spline projector $P_{\boldsymbol{x}^*} : f \to \mathcal{S}_k(\Delta)$ with the norm bounded independently of Δ.*

Proof. For any Δ, take x^* as the points of equioscillation of the Chebyshev spline $s_* =: \sum_i a_j^* N_j$. Then

$$\|P_{\boldsymbol{x}^*}\|_\infty \le \|A_*^{-1}\|_\infty \le \|a^*\|_\infty \, .$$

By previous lemma, $|a_i^*| = \|\lambda_i\|$, where (λ_i) are functionals dual to (N_j), and we know that their norms are bounded by c_k independently of Δ, hence

$$\|a^*\|_\infty = \max_i |a_i^*| = \max_i \|\lambda_i\|_\infty \le c_k. \tag{20.4}$$

Remark 20.6 Thus, given a knot sequence Δ there exists a sequence $\boldsymbol{x}^*$ of 'good' interpolating points that provides a bounded spline interpolation. The converse problem has, however, a negative solution: for any M there exists $\boldsymbol{x}$ such that, no matter how we choose Δ, the projector $P_{\boldsymbol{x}} : f \to \mathcal{S}_k(\Delta)$ has a large norm $\|P_{\boldsymbol{x}}\| \ge M$.

The next theorem gives an estimate for c_k in (20.4), hence a bound for the norm of Demko's projector (and for some other application).

Theorem 20.7 (Scherer, Shadrin [1999]) *For any k, and any Δ, we have*

$$\max_i \|\lambda_i\|_\infty \le k\, 2^k$$

20.2 Exercises

20.1. Prove that

$$\|\lambda_i\| = \frac{1}{\operatorname{dist}(N_i, \operatorname{span}_{j\neq i} N_j)}.$$

20.2. *B-spline basis condition number.* The condition number of a basis V of an n-dimensional normed linear space $\mathbb{X}$ is the value

$$\kappa(V) := \sup_a \frac{\|a\|_\infty}{\|\sum_i a_i v_i\|} \cdot \sup_b \frac{\|\sum_i b_i v_i\|}{\|b\|_\infty}.$$

It measures the extent to which relative changes in the coordinates a of an element $Va := \sum_i a_i v_i$ may be close to the relative change in Va itself.

Alternatively, $\kappa(V) := \frac{c_2}{c_1}$ where c_1, c_2 are the *best* constants in the inequalities

$$c_1 \|a\|_\infty \leq \|\textstyle\sum_i a_i v_i\| \leq c_2 \|a\|_\infty.$$

Prove that for the B-spline basis $B = (N_i)$

$$\kappa(B) = \max_i \|\lambda_i\|.$$

Hint. Show that $c_2 = 1$.

20.3. Prove that, for any sequence $y = (y_i)_{i=1}^k$ of positive numbers, there is a polynomial $p \in \mathcal{P}_{k-1}$ which has y_i as the values of its local extrema. Precisely, there is a sequence

$$0 = x_1 < x_2 < \cdots < x_k = 1$$

such that

$$p(x_i) = (-1)^i y_i, \qquad p'(x_i) = 0, \quad i = 2..k-1.$$