**Part III - Lent Term 2004**

# Approximation Theory – Lecture 22

## 22 Orthogonal spline projector

### 22.1 Error bounds for spline interpolation

**Discussion 22.1** Let $f \in W_2^k[a,b]$, and, for a given knot-sequence $\Delta = (t_i)$, let $s$ be the interpolating spline of degree $2k-1$ to $f$ on $\Delta$

$$s(t_i) = f(t_i), \qquad s \in \mathcal{S}_{2k}(\Delta).$$

Then $\sigma := s^{(k)}$ is the best $L_2$-approximation to $f^{(k)}$ from the spline space $\mathcal{S}_k(\Delta)$, in particular, the following inequalities are valid

$$(f^{(k)} - s^{(k)}, s^{(k)}) = 0, \qquad \|f^{(k)} - s^{(k)}\| \le \|f^{(k)}\|, \qquad \|s^{(k)}\| \le \|f^{(k)}\|.$$

The last inequality is the minimal norm property of the interpolating spline $s$, the middle one is useful in deriving Jackson-type estimates for the error of spline-interpolation.

**Lemma 22.2** *Let $f \in C[a,b]$, and let $\ell \in \mathcal{P}_{k-1}$ be the Lagrange interpolating polynomial to $f$ on $\delta = (t_1, ..., t_k)$. Then, for $x \in [t_1, t_k]$,*

$$f(x) - \ell(x) = \omega(x) f[x, t_1, ..., t_k] = \frac{1}{k!}\omega(x) \int_{t_1}^{t_k} M_x(t) f^{(k)}(t)\, dt\,,$$

*where $M_x$ is the B-spline of degree $k-1$ with the knot sequence $(x, t_1, ..., t_k)$*

**Proof.** The last equality is the known property of divided differences, so we need to prove the first one. For $x \in \delta$ it holds trivially, for the remaining case, let $\bar{x}$ be any other point from $[a,b]$, and let $\widetilde{\ell} \in \mathcal{P}_k$ be the Lagrange polynomial that interpolates $f$ on $(\bar{x}, t_1, ..., t_k)$. Then

$$\widetilde{\ell}(x) - \ell(x) = \omega(x) f[\bar{x}, t_1, ..., t_k]\,.$$

Indeed, the left-hand side is a polynomial of degree $k$ that vanishes at $(t_1 ... t_k)$, and its leading coefficient is the corresponding divided difference. To finish the proof, we take $x = \bar{x}$. ∎

**Remark 22.3** Denote by $W_2^k(\Delta)$ the subclass of functions in $W_2^k[a,b]$ that vanish on $\Delta$. Then the previous lemma is equivalent to the statement that, for $f \in W_2^k(\Delta)$, we have the estimate

$$f(x) = \frac{1}{k!}\,\omega(x) \int_{t_i}^{t_{i+k}} M_x(t) f^{(k)}(t)\, dt\,, \quad x \in [t_i, t_{i+k}]\,. \tag{22.1}$$

**Lemma 22.4** *Let $f \in W_2^k(\Delta)$. Then*

$$|f(x)| \le c_k |t|^{k-1/2} \|f^{(k)}\|_{L_2[t_i, t_{i+k}]} \quad x \in [t_i, t_{i+k}]\,. \tag{22.2}$$

**Proof.** First, we apply the Cauchy-Schwartz inequality to the integral in the right-hand side of (22.1), and the trivial estimate for $|\omega(x)|$:

$$|f(x)| \le c_k |\omega(x)| \|M_x\|_2 \|f^{(k)}\|_2 \le c_k' |t|^k \|M_x\|_2 \|f^{(k)}\|_2\,.$$

Then we use the Hölder inequality to evaluate the $L_2$-norm of $M_x$. We have $M_x(t)^2 = M_x(t) M_x(t) = \frac{k}{t_{i+k} - t_i} N_x(t) M_x(t)$, whence

$$\|M_x\|_2 \le c_k |t|^{-1/2} \|N_x\|_\infty^{1/2} \|M_x\|_1^{1/2} \le c_k |t|^{-1/2}\,.$$

**Theorem 22.5** *Let $f \in W_2^k[a,b]$, and, for a given knot-sequence $\Delta = (t_i)$, let $s$ be the interpolating spline of degree $2k-1$ to $f$ on $\Delta$. Then*

$$\|f - s\|_\infty \le c_k |t|^{k-1/2} \|f^{(k)}\|_2 \,.$$

**Proof.** We use the previous local estimate for the functions vanishing on a certain grid and the orthogonal properties of interpolating spline. Given $x$, we choose any $i$ such that $x \in [t_i, t_{i+k}]$ to obtain

$$|f(x) - s(x)| \le c_k\, |t|^{k-1/2}\, \|f^{(k)} - s^{(k)}\|_{L_2[t_i, t_{i+k}]} \,,$$

and then

$$\|f^{(k)} - s^{(k)}\|_{L_2[t_i,t_{i+k}]} \le \|f^{(k)} - s^{(k)}\|_{L_2[a,b]} \le \|f^{(k)}\|_{L_2[a,b]} \,.$$

**Discussion 22.6** As we know, for smooth functions, the order of best approximation by splines of degree $k-1$ is $|t|^k$. So, the question is whether interpolating splines of degree $2k-1$ could provide the approximation order higher then $|t|^{k-1/2}$, in particular, whether the maximal order $|t|^{2k}$ can be achieved, say, for the functions which are $2k$ times continuously differentiable. The next theorem shows a simple way to increase the order up to $|t|^{2k-1}$.

**Theorem 22.7** *Let $f \in W_1^{2k}[a,b]$, and let $s \in \mathcal{S}_{2k}(\Delta)$ be the interpolating spline with the boundary conditions*

$$s^{(r)}(a) = f^{(r)}(a), \quad s^{(r)}(b) = f^{(r)}(b), \quad r = 0...k-1 \tag{22.3}$$

*(which correspond to the knot-sequence with $t_1 = \cdots = t_k = a$ and $b = t_{n+1} = \cdots = t_{n+k}$). Then*

$$\|f - s\|_\infty \le c_k\, |t|^{2k-1}\, \|f^{(2k)}\|_1 \,. \tag{22.4}$$

**Proof.** We start with the estimate

$$\|f - s\|_\infty \le c_k\, |t|^{k-1/2}\, \|f^{(k)} - s^{(k)}\|_2 \,, \tag{22.5}$$

and what we want is some further estimate, in terms of a higher derivative of $f$, for the right-hand side, which is actually the best $L_2$-approximation of $f^{(k)}$ by $\mathcal{S}_k(\Delta)$. Such estimate can be derived using quasi-interpolants, but there is a simpler way to do it which is based solely on the properties of interpolating spines.

We transform the right-hand side of (22.5) by integration by parts:

$$\begin{aligned}
\|f^{(k)} - s^{(k)}\|_2^2 &= \int_a^b [f^{(k)} - s^{(k)}] f^{(k)}(t)\, dt = -\int_a^b [f^{(k-1)}(t) - s^{(k-1)}(t)] f^{(k+1)}(t)\, dt \\
&= \cdots = (-1)^k \int_a^b [f(t) - s(t)] f^{(2k)}(t)\, dt \le \|f - s\|_\infty \|f^{(2k)}\|_1 \,.
\end{aligned}$$

Here the first equality is due to the orthogonality property of splines, and all non-integral terms vanish because of the boundary conditions (22.3). Now we may apply the initial estimate (22.5) to the factor $\|f - s\|_\infty$ on the right-hand side to obtain

$$\|f^{(k)} - s^{(k)}\|_2^2 \le c_k\, |t|^{k-1/2}\, \|f^{(k)} - s^{(k)}\|_2 \|f^{(2k)}\|_1 \,,$$

whence, on deleting the common factor,

$$\|f^{(k)} - s^{(k)}\|_2 \le c_k\, |t|^{k-1/2}\, \|f^{(2k)}\|_1 \,.$$

Combining this inequality with (22.5), we arrive at (22.4).

**Remark 22.8** A similar technique allows to get the $\mathcal{O}(|t|^{2k-1/2})$-estimate for the functions $f$ from $W_2^{2k}$,

$$\|f - s\|_\infty \le c_k\, |t|^{2k-1/2}\, \|f^{(2k)}\|_2 \,,$$

but it does not provide further improvements for $f \in W_p^{2k}$ if $p > 2$, in particular for $p = \infty$. This can be fixed if we manage to prove that orthogonal spline projector is bounded as an operator from $C$ to $C$.

## 22.2 Exercises

**22.1.** Let $G$ be the class of functions $g$ in $W_2^k[a,b]$ such that

$$g^{(r)}(a) = g^{(r)}(b) = 0, \quad r = 0...k-1\,.$$

Prove that $f \in L_2[a,b]$ is orthogonal on $[a,b]$ to all polynomials of degree $k-1$ if and only if $f = g^{(k)}$ for some $g \in G$.

**22.2.** Lemma 22.2 provide us with the sharp $L_p$-bound of the Lagrange interpolation

$$\|f - \ell\|_{L_p[-1,1]} \le \frac{1}{k!}\|\omega\|_{L_p[-1,1]}\|f^{(k)}\|_\infty\,.$$

As in the case of the uniform approximation, we may look for the best choice of interpolation knots which minimize $\|\omega\|_p$. This leads to the problem of finding the polynomial of least $L_p$ deviation from zero,

$$\|x^k + a_{k-1}x^{k-1} + \cdots + a_0\|_{L_p[-1,1]} \to \inf_a\,.$$

Prove that, for the $L_2$-norm, solution is given by the Legendre polynomial

$$L_k(x) = \text{const}\,\frac{d^k}{dx^k}(x^2-1)^k\,.$$