Part III - Lent Term 2004

Approximation Theory – Lecture 23

23 Orthogonal spline projector

23.1 Max-norm of the L_2-spline projector

Definition 23.1 Denote by $P_{\mathcal{S}}$ the operator of orthoprojection onto $\mathcal{S}$ with respect to the ordinary inner product $(f,g) := \int_a^b fg$,

$$P_{\mathcal{S}} : C[a,b] \to \mathcal{S}_k(\Delta), \qquad (P_{\mathcal{S}} f, \sigma) = (f, \sigma) \quad \forall \sigma \in \mathcal{S}_k(\Delta). \tag{23.1}$$

We are interested in the max-norm of $P_{\mathcal{S}}$,

$$\|P_{\mathcal{S}}\|_\infty := \sup \frac{\|P_{\mathcal{S}} f\|_\infty}{\|f\|_\infty},$$

that provides us with the bound $\|f - P_{\mathcal{S}} f\|_\infty \le (1 + \|P_{\mathcal{S}}\|_\infty) E_{k,\Delta}(f)$.

Method 23.2 With respect to the max-norm, we change slightly Method 21.4. For $f \in C[a,b]$ we set $P_{\mathcal{S}} f = \sum a_j N_j$, but we run σ in (23.1) through the B-splines M_i, i.e.,

$$(\textstyle\sum_{j=1}^n a_j N_j, M_i) = (f, M_i), \quad i = 1, ..., n.$$

The outcoming linear system for determining the coefficients (a_i) looks as follows

$$Ga = b, \qquad G = (M_i, N_j), \quad b_i := (f, M_i).$$

Lemma 23.3 *For any $k, n \in \mathbb{N}$ and any Δ, we have the bound*

$$\|P_{\mathcal{S}}\|_\infty \le \|G^{-1}\|_\infty .$$

Proof. As before,

$$\|P_{\mathcal{S}} f\|_\infty = \|\textstyle\sum_i a_j N_j\|_\infty \le \|a\|_\infty ,$$

and since $a = G^{-1} b$, whereas $\|b\|_\infty := \max_i |(f, M_i)| \le \|f\|_\infty \|M_j\|_1 = \|f\|_\infty$, we obtain

$$\|a\|_\infty \le \|G^{-1}\|_\infty \|b\|_\infty \le \|G^{-1}\|_\infty \|f\|_\infty.$$

Thus, $\|P_{\mathcal{S}} f\|_\infty \le \|G^{-1}\|_\infty \|f\|_\infty$ for all f, and that implies the result. ∎

So, we need some tools to evaluate the Gramian inverse. The next statement relates the elements of G^{-1} to the elements of dual basis.

Definition 23.4 (Dual basis) The *dual basis* for (u_i) is the sequence $(u_i^*)_{i=1}^n$ that satisfies

$$u_i^* \in \mathcal{U}_n, \quad (u_i^*, u_j) := \delta_{ij}.$$

Lemma 23.5 *Let $G := [(u_i, u_j)]_{i,j=1}^n$ be the Gram matrix for (u_i) and let $u_k^* =: \sum_{j=1}^n b_{jk} u_j$. Then*

$$G^{-1} = (b_{ij}),$$

i.e., the columns of G^{-1} are the coefficients of expansion of the elements of the dual basis with respect to the original basis.

Proof. Set $H := (b_{ij})$. Then $(GH)_{ik} = \sum_j (u_i, u_j) b_{jk} = (u_i, \sum_j b_{jk} u_j) = (u_i, u_k^*) = \delta_{ik}$, i.e., $H = G^{-1}$. ∎

Remark 23.6 We cannot say much more about general Gramian (which could be any positive definite matrix), so we need to exploit some specific properties of B-splines. First of all, we define the L_2-normalized B-spline basis as

$$\widehat{N}_i := M_i^{1/2} N_i^{1/2} = (\tfrac{k}{|J_i|})^{1/2} N_i = (\tfrac{k}{|J_i|})^{-1/2} M_i \qquad J_i := t_{i+k} - t_i.$$

Then there exists a number d_k, called the B-spline basis condition number, such that

$$d_k^{-1} \|a\|_{\ell_2} \le \|\textstyle\sum_i a_i \widehat{N}_i\|_{L_2} \le \|a\|_{\ell_2}.$$

23.2 Exponential decay of the Gramian inverse

Definition 23.7 By $\mathcal{S}_{k,d}$ we denote the set of bases $(u_i)_{i=1}^n$ in $L_2[a,b]$ such that

$$\begin{array}{ll} 1) & \operatorname{supp} u_i = [t_i, t_{i+k}] \text{ for some increasing knot-sequence } (t_i)_{i=1}^{n+k} \qquad (23.2) \\ 2) & d^{-2}\|a\|_{\ell_2}^2 \le \|\sum_i a_i u_i\|_{L_2}^2 \le \|a\|_{\ell_2}^2. \qquad (23.3) \end{array}$$

The number d is called the basis condition number.

Proposition 23.8 *Let (u_i) be an $\mathcal{S}_{k,d}$-basis and let $u_i^* = \sum_{j=1}^n b_j u_j$. Then*

$$|b_j| \le d^2 \gamma_0^{|i-j|}, \quad \gamma_0 = \left(\frac{d^2-1}{d^2+1}\right)^{1/(2k-2)} < 1. \tag{23.4}$$

Proof. 1) Since u_i^* is orthogonal to $\operatorname{span}(u_j)_{j\ne i}$, we have

$$(u_i^*, u_i^*) = (u_i^*,\, b_i u_i + \textstyle\sum_{j\ne i} b_j u_j) = (u_i^*, b_i u_i) = b_i,$$

i.e., $\|u_i^*\|_2^2 = b_i$. But, by (23.3), $d^{-2}\sum_{j=1}^n b_j^2 \le \|u_i^*\|_2^2$, so $\sum_{j=1}^n b_j^2 \le d^2 b_1$ and we derive

$$\textstyle\sum_{j=1}^n b_j^2 \le d^4. \tag{23.5}$$

2) Consider the case $j > i+1-k$. For $m > i+1-k$, set

$$v_m := \textstyle\sum_{j=m}^{n+k} b_j u_j.$$

Then $v_m(x)$ coincides with $u_i^*(x)$ for $x \ge t_{m+k-1}$, therefore v_m is orthogonal to $\operatorname{span}(u_j)_{j\ge m+k-1}$, in particular to v_{m+k-1}. Thus, by Pythagoras theorem,

$$\|v_m\|_{L_2[a,b]}^2 + \|-v_{m+k-1}\|_{L_2[a,b]}^2 = \|v_m - v_{m+k-1}\|_{L_2[a,b]}^2,$$

or,

$$\|\textstyle\sum_{j\ge m} b_j u_j\|_2^2 + \|\sum_{j\ge m+k-1} b_j u_j\|_2^2 = \|\sum_{j=m}^{m+k-2} b_j u_j\|_2^2.$$

One more use of (23.3) implies

$$d^{-2}\textstyle\sum_{j\ge m} b_j^2 \;+\; d^{-2}\sum_{j\ge m+k-1} b_j^2 \;\le\; \sum_{j=m}^{m+k-2} b_j^2 \tag{23.6}$$

3) Set

$$B_m := \textstyle\sum_{j\ge m} b_j^2. \tag{23.7}$$

Then (23.6) reads

$$d^{-2}B_m + d^{-2}B_{m+k-1} \le B_m - B_{m+k-1} \tag{23.8}$$

or

$$B_{m+k-1} \le \gamma_0 B_m, \quad \gamma_0 := \frac{d^2-1}{d^2+1}.$$

For any $j \ge i+1$, with some $0 \le s \le k-2$, this gives

$$b_j^2 \le B_j \le \gamma^{\lfloor \frac{j-i}{k-1}\rfloor + 1} B_{i-s} \le \gamma^{\frac{j-i}{k-1}} d^4,$$

the latter estimate by (23.5). Finally,

$$|b_j| \le d^2\gamma_0^{|i-j|}, \quad \gamma_0 = \left(\frac{d^2-1}{d^2+1}\right)^{1/(2k-2)}.$$

Corollary 23.9 *Let (u_i) be an $\mathcal{S}_{k,d}$-basis and let G be its Gram matrix. Then*

$$\|G^{-1}\|_\infty \le c_{k,d}.$$

Proof. By Lemma (23.5), $G^{-1} = (b_{ij})$, and by (23.4) the values $|b_{ij}|$ decay exponentially away from the diagonal, thus

$$\|G^{-1}\|_\infty = \max_i \textstyle\sum_j |b_{ij}| \le c_{k,d}.$$

23.3 Exercises

23.1. Prove that if $G = [(u_i, u_j)]$ then the elements of the Gramian inverse are

$$(G^{-1})_{ij} = (u_i^*, u_j^*).$$

23.2. Prove that the value of the least squares approximation to $f \in \mathbb{X}$ from $\mathcal{U}_n =^(u_i)$ is given by

$$E(f)^2 = \frac{G(f, u_1, ..., u_n)}{G(u_1, ..., u_n)},$$

where Gs are the corresponding Gramians.

23.3. (*Inverse of a band matrix.*) An exponential decay of the Gramian inverse (or, equivalently, of the norms of dual elements) was proved originally for the B-spline basis. Its generalization to the finitely supported and well-conditioned bases (as given in Proposition 23.8) appeared by analyzing which properties of the spline function are actually involved in the proof. Subsequently, Demko eliminated even the bases and reduced the statement to the pure fact about band matrices.

Demko theorem. Let A be a band matrix, that is bounded and has a bounded inverse, i.e.,

$$1') \quad a_{ij} = 0 \quad \text{if} \quad |i-j| \ge k.$$
$$2') \quad \|A\|_2 = c_1, \quad \|A^{-1}\|_2 = c_2.$$

Then the elements of $A^{-1} = (b_{ij})$ decays exponentially away from the diagonal, i.e.,

$$|b_{ij}| \le c\,\gamma^{|i-j|}, \quad \gamma < 1,$$

where c and γ depend on c_1, c_2 and k. (For you to prove.)

Remark. This is how the Gramian $G = (u_i, u_j)$ of the $\mathcal{S}_{k,d}$-basis fits into Demko theorem:

1) Finiteness of support (23.2) provides exactly (1′)

2) The basis condition (23.3) is the inequality $d^{-2}\|x\|^2 \le (Gx, x) \le \|x\|^2$ which implies

$$d^{-2} \le \lambda_{\min}(G) = \|G^{-1}\|_2^{-1}, \quad \|G\|_2 = \lambda_{\max}(G) \le 1.$$

Hint. Try to obtain analogues of (23.6)-(23.7) by considering the vectors

$$b^{(m)} = (0, ..., 0, b_m, b_{m+1}, ..., b_n)$$

and by showing that, for some $r = r(k)$, the vectors $Ab^{(m)}$ and $Ab^{(m+r)}$ have non-overlapping supports. (Support of a vector is a sequence of the coordinates of its non-zero components. If supports of x, y donot overlap, then $\|x \pm y\|^2 = \|x\|^2 + \|y\|^2$).

You may also find useful the inequalities

$$\|Ax\| \le \|A\|\,\|x\|, \qquad \|Ax\| \ge \|A^{-1}\|^{-1}\,\|x\|.$$