Part III - Lent Term 2004

Approximation Theory – Lecture 24

24 Orthogonal spline projector

24.1 Max-norm of the L_2-spline projector on the quasi-uniform meshes

Theorem 24.1 (Douglas, Dupont, Wahlbin [1976]) *If the knot-sequence Δ is quasi-uniform, i.e., the global mesh ratio $M := \max_{i,j} \frac{t_{i+1}-t_i}{t_{j+1}-t_j}$ is bounded, then*

$$\|P_{\mathcal{S}_k(\Delta)}\|_\infty < c_k M^{1/2},$$

i.e., the orthoprojector $P_{\mathcal{S}_k(\Delta)}$, as an operator from $C[a,b]$ onto $C[a,b]$, is bounded.

Proof. Let

$$\widehat{N}_i = \Big(\tfrac{k}{|J_i|}\Big)^{1/2} N_i = \Big(\tfrac{|J_i|}{k}\Big)^{1/2} M_i, \quad \widehat{G} := (\widehat{N}_i, \widehat{N}_j), \quad G := (M_i, N_j).$$

The B-spline basis $(\widehat{N}_i)$ satisfies (23.2)-(23.3), i.e., it is finitely supported and good conditioned. Therefore, we may apply Corollary 23.9 to the Gram matrix $\widehat{G}$ to obtain $\|\widehat{G}^{-1}\|_\infty < c_k$. We need however an estimate for $\|G^{-1}\|_\infty$. By definition, $(M_i, N_j) = (\frac{k}{|J_i|})^{1/2}(\widehat{N}_i, \widehat{N}_j)(\frac{k}{|J_j|})^{-1/2}$, so with $D := \operatorname{diag}(|J_1|, ..., |J_n|)$ we have the relations

$$G = D^{-1/2}\widehat{G}D^{1/2} \quad \text{hence} \quad G^{-1} = D^{-1/2}\widehat{G}^{-1}D^{1/2}.$$

It follows that $b_{ij} = \widehat{b}_{ij}(|J_i|/|J_j|)^{-1/2}$, therefore $|b_{ij}| \le M^{1/2}\,|\widehat{b}_{ij}|$, and

$$\|P_{\mathcal{S}}\|_\infty \le \|G^{-1}\|_\infty \le M^{1/2}\|\widehat{G}^{-1}\|_\infty \le M^{1/2}c_k\,. \qquad \blacksquare$$

Corollary 24.2 *Let $f \in C^k$ and let $s \in \mathcal{S}_{2k}(\Delta)$ be the interpolating spline to f on Δ. If the knot-sequence Δ is quasi-uniform, then*

$$\|f - s\|_\infty \le c_k\, M^{1/2}|t|^k\, E_{k,\Delta}(f^{(k)})_\infty\,.$$

In particular, for $f \in C^{2k}$, we obtain the optimal order of approximation $\|f - s\|_\infty \le c_k|t|^{2k}\|f^{(2k)}\|_\infty$.

Discussion 24.3 Using the bounds for the max-norm of the orthogonal spline projector as the intermediate tools for bounding the error of spline interpolation caused certain, rather severe, restrictions on the cases considered. Namely, we required from f to be at least k times differentiable, and we required from Δ to be quasi-uniform. A natural question is whether these two assumptions could be somehow relaxed. And the answer is that having one of these assumptions, there is no need for another. What we are going to prove next is that, for the quasi-uniform meshes Δ, the interpolating projector $P : C \to \mathcal{S}_{2k}(\Delta)$ is bounded as an operator from C to C. What we are not going to prove is that, for k (and more) times differentiable functions f, the interpolating splines converge to f with the optimal order independently of the knot-sequence. It is, however, the technique developed for the orthogonal projector that allows to make both conclusions.

Lemma 24.4 *Let $s_i \in S_{2k}(\Delta)$ be the fundamental interpolating spline, i.e., $s_i(t_j) = \delta_{ij}$, and let $s_i^{(k)} = \sum_j b_j \widehat{N}_j$ be expansion of its k-th derivative. Then*

$$|b_j| \le c_k \gamma_k^{|i-j|}\|b\|_{\ell_2}$$

Proof. The proof is parallel to that of Proposition 23.8. If $j > i$, then $(s^{(k)}, \widehat{N}_j) = c\, s_i[t_j ... t_{j+k}] = 0$, i.e., $s_i^{(k)}$ is orthogonal to all B-splines $(\widehat{N}_j)_{j>i}$. Setting $v_m := \sum_{j=m}^n b_j \widehat{N}_j$ we obtain that v_m coincides with s_i on $[t_{m+k-1}, t_n]$, hence it is orthogonal to v_{m+k-1}, etc.

Corollary 24.5 *For any Δ, the L_2-norm of $s_i^{(k)}$ on subintervals decays exponentially away from t_i,*

$$\|s_i^{(k)}\|_{L_2[t_j,t_{j+k}]} \le c_k\,\gamma_k^{|i-j|}\,\|s_i^{(k)}\|_{L_2[a,b]}\,. \tag{24.1}$$

Proof. We have $\|s_i^{(k)}\|_2 \ge \frac{1}{d}\,\|b\|_{\ell_2}$ and $\|s_i^{(k)}\|^2_{L_2[t_j,t_{j+k}]} = \|\sum_{j-k+1}^{j+k-1} b_m \widehat{N}_m\|^2_{L_2[t_j,t_{j+k}]} \le \sum_{j-k+1}^{j+k-1} b_m^2$.

Corollary 24.6 *If Δ is quasi-uniform, then the max-norm of s_i on subintervals decays exponentially away from t_i,*

$$\|s_i\|_{C[t_j,t_{j+k}]} \le c_k\,M^{k-1/2}\gamma_k^{|i-j|}\,, \quad M := \max\frac{t_{j+1}-t_j}{t_{i+1}-t_i}$$

Proof. On combining the estimate (22.1), which is valid for any function with given zeros, with (24.1) we obtain

$$\|s_i^{(k)}\|_{C[t_j,t_{j+k}]} \le c_k|t|^{k-1/2}\,\|s_i^{(k)}\|_{L_2[t_j,t_{j+k}]} \le c_k'|t|^{k-1/2}\gamma_k^{|i-j|}\|s_i^{(k)}\|_{L_2[a,b]}\,,$$

and we are done once we prove that

$$\|s_i^{(k)}\|_{L_2[a,b]} \le c_k\,|t_{i\pm1}-t_i|^{-(k-1/2)}\,. \tag{24.2}$$

Let f be any "fundamental" function from $W_2^k[a,b]$ such that $f(t_j)=\delta_{ij}$. Then s_i interpolates f on Δ, and by the minimal norm property $\|s_i^{(k)}\|_2 \le \|f^{(k)}\|_2$. So, it is sufficient to prove (24.2) for any particular "fundamental" f. The simplest choice is a function f supported in $[t_{i-1},t_{i+1}]$, for example, take $f(x)=F(\frac{x-t_i}{h})$, where $F(x)=(1-x^2)_+^k$ and $h=\min|t_{i\pm1}-t_i|$.

Theorem 24.7 *If Δ is quasi-uniform, then the interpolating projector P, as an operator from $C[a,b]$ to $C[a,b]$, is bounded, in particular*

$$\|f-s(f)\| \le c_k(M)E_{2k,\Delta}(f)_\infty\,.$$

Proof. We have $\|P\|_\infty = \|L\|_\infty$ where $L(x)=\sum_i|s_i(x)|$ is the Lebesgue function of the interpolation projector. Now, if $x\in[t_j,t_{j+1}]$, then

$$\textstyle\sum_i|s_i(x)| \le c_k(M)\sum_i\gamma^{|i-j|} \le c_k'(M).$$

Remark 24.8 There was a famous de Boor's conjecture of 1972 that the max-norm of the orthogonal spline projector is bounded independently of Δ.

Theorem 24.9 (Shadrin [2001]) *For any k and any Δ,*

$$\|P_{\mathcal{S}_k(\Delta)}\|_\infty < c_k.$$

24.2 Exercises

24.1. A matrix $A \in \mathbb{R}^{n\times n}$ is called *strictly diagonally dominant* by rows if

$$|a_{ii}| - \sum_{j\neq i} |a_{ij}| \geq \gamma > 0 \quad i = 1, ..., n. \tag{1}$$

Example: $A = \begin{bmatrix} 4 & 1 & \\ 1 & 4 & 1 \\ & 1 & 4 \end{bmatrix}$. Prove that, if (1) holds, then $\|A^{-1}\|_\infty \leq 1/\gamma$.

Hint. Use equality $\|A^{-1}\|_\infty = \sup_x \frac{\|x\|_\infty}{\|Ax\|_\infty}$.

24.2. 1) For $k = 2$ and for any Δ, compute the entries of $G = (M_i, N_j)$.

Hint. This Gramian is tri-diagonal and 'shift-invariant' by rows in the sense that the next row is obtained from the previous one by the shift $t_i \to t_{i+1}$. Hence only three elements $g_{i,i-1}$, $g_{i,i}$ and $g_{i,i+1}$ need to be computed. Notice that the sum of elements in any row must be equal to one, since

$$\sum_{j=1}^{n} g_{ij} = \sum_{j=1}^{n} (M_i, N_j) = (M_i, \sum_{j=1}^{n} N_j) = (M_i, 1) = 1.$$

2) Prove the theorem by Ciesielski [1963]: For $k = 2$ and for *any* Δ,

$$\|P_{\mathcal{S}}\|_\infty \leq 3.$$