Part III - Lent Term 2002

Approximation Theory – Exercise Sheet 1

1.1. Prove that the norm of a linear projector M satisfies $\|M\| \ge 1$. By considering the linear interpolation $M : \mathbb{C}[a,b] \to \mathcal{P}_1$ at the end-points of $[a,b]$ show that, for the linear projectors, both inequalities

$$\|M\| \ge 1, \qquad \|f - Mf\| \le (\|M\| + 1)\,\mathrm{dist}(f, \mathcal{A})$$

may become equalities (i.e., they are *sharp*).

1.2. An approximation $M : \mathbb{X} \to \mathcal{A}$ is called *near-best*, if there is a constant $\gamma > 0$ such that $\|f - Mf\| \le \gamma\,\mathrm{dist}(f, \mathcal{A})$, $\forall f \in \mathbb{X}$. Prove that the near-best approximation M is a projection.

1.3. Let $M : \mathbb{C}[a,b] \to \mathcal{P}_n$ be the interpolation at some given points $a \le x_0 < \cdots < x_n \le b$, and let $\ell_i = \prod_{j \ne i} \frac{x - x_j}{x_i - x_j}$ be the corresponding fundamental polynomials. The function

$$L(x) := \textstyle\sum_{i=0}^n |\ell_i(x)|$$

is called the *Lebesgue* function (so that, from Example 1.7, $\|L\|_\infty = \|M\|_\infty$). Prove that, on $[x_k, x_{k+1}]$, the Lebesgue function L coincides with the polynomial $q \in \mathcal{P}_n$ such that

$$|q(x_i)| = 1 \quad \text{all} \quad i, \qquad q(x_k) = q(x_{k+1}) = 1, \qquad q(x_i) = -q(x_{i+1}) \quad \text{for} \quad i \ne k.$$

1.4. For the quadratic interpolation M, find all positions of the interpolation points that satisfy $-1 \le x_0 < x_1 < x_2 \le 1$ and that provide the value $\|M\|_\infty = 5/4$.

2.1. Using Weierstrass theorem prove that the polynomials are dense in $C^k[0,1]$, the Banach space of all k times continuously differentiable functions on $[0,1]$ with the norm

$$\|f\|_\infty^{(k)} := \max_{j \le k} \|f^{(j)}\|_\infty\,.$$

2.2. Prove that, for any positive linear operator U, we have $\|U\| = \|U(1,\cdot)\|$. Then derive that, under assumption of Korovkin theorem,

$$\sup_n \|U_n\| < M < \infty.$$

(The latter inequality is in fact a *necessary* condition for any sequence (U_n) of linear operators to provide convergence $U_n(f) \to f$ for *all* f in $\mathbb{C}[a,b]$.

3.1. Prove: If $f \in C[0,1]$ vanishes at 0 and 1, then the sequence $(B_n^* f)$,

$$B_n^*(f, x) := \textstyle\sum_{k=1}^{n-1} \left\lfloor \binom{n}{k} f\left(\frac{k}{n}\right) \right\rfloor x^k (1-x)^{n-k},$$

which consists of polynomials with *integer* coefficients in the standard *power* form, converges uniformly to f. (Here $\lfloor t \rfloor$ is the largest integer not bigger than t.)

3.2. Prove: If $f \in C^k[0,1]$, then not only does $B_n(f)$ converge uniformly to f, but also the k-th derivatives of $B_n(f)$ converge uniformly to $f^{(k)}$.

Hint. You may use the fact that $h^{-k} \Delta_h^k(f, t) = f^{(k)}(\xi)$ with some $\xi \in [t, t + kh]$.

3.3. Find the explicit expression for $B_n(p_1)$ and $B_n(p_2)$ where $p_\ell(x) = x^\ell$ directly from the formula (3.1). *Hint.* Differentiate the identity

$$\textstyle\sum_{k=0}^n \binom{n}{k} p^k q^{n-k} = (p+q)^n$$

with respect to p.

3.4. For the Fejer kernel F_n, prove the formula (3.3), and that $\|F_n\|_{L_1[-\pi,\pi]} = 1$. Prove that the Fejer operator $\sigma_n : C(\mathbb{T}) \to C(\mathbb{T})$ satisfies $\|\sigma_n\| = 1$. Why is the Lebesgue inequality

$$\|f - \sigma_n(f)\| \le (\|\sigma_n\| + 1)E_n(f) = 2E_n(f)$$

not applicable for the linear operator σ_n?

4.1. Prove that the equality $\|x\| = \|y\| = \|\frac{1}{2}x + \frac{1}{2}y\| = 1$ implies that $\|\alpha x + (1-\alpha)y\| = 1$ for any $\alpha \in [0,1]$, i.e., that two conditions in Definition 4.4 are really equivalent.

4.2. Does the unicity Theorem 4.6 have a converse? That is, can we infer strict convexity from a knowledge that for each $f \in \mathbb{X}$ and for each finite-dimensional $A \subset \mathbb{X}$ there exists a *unique* element of best approximation?

Hint. Let $\mathbb{X} = \mathbb{R}^2$ with a norm which is not strictly convex, i.e., the set $\|x\| = 1$ contains a line segment. Find f and one-dimensional subspace $\mathcal{A}_1$ with several b.a. to f.

4.3. Fill in the details of the following proof of Theorem 4.2. It is based on the Weierstrass theorem: *A continuous function on a closed bounded set in $\mathbb{R}^n$ attains its minimum.*

a) Let $f \in \mathbb{X}$, an let $\mathcal{A}_n = \operatorname{span}(g_k)_{k=1}^n$ where g_k are linearly indepdendent.

b) For $x = (x_1, \ldots, x_n)$, the function $G(x) := \|\sum_{k=1}^n x_k g_k\|$ is a continuous function of n real variables x_k, hence, on the unit ball $\sum x_k^2 = 1$, it takes some minimal value $\rho > 0$.

c) The function $F(x) := \|f - \sum_{k=1}^n x_k g_k\|$ is continuous as well, and it follows from (b) that $F(x) \to \infty$ as $x \to \infty$.

d) Thus, looking for the infimum of $F(x)$ over $x \in \mathbb{R}^n$, we may restrict ourselves to some ball $B_r := \{x : \sum x_k^2 \le r\}$, this is a bounded closed set in $\mathbb{R}^n$, hence, on B_r, $F(x)$ attains its minimal value.

4.4. The essence of the example given in Lemma 4.3 is that the linear functional

$$\lambda : c_0 \to \mathbb{R}, \quad \lambda(f) := \textstyle\sum_{k=1}^\infty 2^{-k}\xi_k, \quad f = (\xi_1, \xi_2, \ldots) \in c_0$$

does not take its norm

$$\|\lambda\| := \sup\{|\lambda(f)| : \|f\| = 1\},$$

i.e., there is no function $f \in c_0$ such that $|\lambda(f)| = \|\lambda\|\,\|f\|$.

a) Prove that, for any $\mathbb{X}$ and any λ in $\mathbb{X}^*$ (the space of all linear bounded functionals on $\mathbb{X}$), we have the equality $|\lambda(f)| = \|\lambda\| \operatorname{dist}(f, \ker\lambda)$ for any $f \in \mathbb{X}$.

b) Using (a), even if you have not proved it, prove that if $\lambda : \mathbb{X} \to \mathbb{R}$ does not take its norm, then the subspace $\mathcal{A}_0 := \ker\lambda$ is a non-existence set.

c) For ℓ_1, the Banach space of infinite sequences f such that

$$f = (\xi_1, \xi_2, \ldots), \quad \textstyle\sum_{k=1}^\infty |\xi_k| < \infty, \quad \|f\| = \sum_k |\xi_k|,$$

construct a functional $\lambda(f) = \sum_{k=1}^\infty \lambda_k \xi_k$ which does not take its norm, thus find a subspace of ℓ_1 which is a non-existence set.

4.5. Derive Krein's Theorem 4.13 from Hobby–Rice Theorem 4.12.

Hint. For any $f \in \mathcal{A}_n$ consider the function $F = |f| \cdot h$, where h is a sign function from Hobby-Rice theorem.

5.1. Prove that $\mathcal{T}_n$, the space of all trig. polynomials of degree $\le n$, is a Haar space on $\mathbb{T} = [-\pi, \pi)$.

Hint. Write $t_n(x) = \sum_{k=-n}^n c_k e^{ikx}$, make the change $z = e^{ix}$, and reduce the problem to the algebraic case.

5.2. Prove that, for distinct λ_k, $\mathcal{A}_n = (e^{\lambda_k x})_{k=0}^n$ is a Haar space on any $[a,b]$.

Hint. For $p(x) = c_0 + \sum_{k=1}^{n+1} c_k e^{\mu_k x}$, $p \in \mathcal{A}_{n+1}$, we have $p' \in \mathcal{A}_n$. Use this fact with induction.

5.3. Prove that Lemma 5.6 is valid for $m = 1$ as well, i.e. in a Haar space $\mathcal{A}_n$ on $[a,b]$ there exists a polynomial q such that $q > 0$.

6.1. For $\mathcal{T}_{n-1}$, the space of trig. polynomials of degree $n-1$, find the value and the polynomial of best approximation to $f(x) = a\cos nx + b\sin nx$.

6.2. Let

$$f(x) = \sum_{k=0}^{\infty} a_k \cos 3^k x, \quad a_k > 0, \quad \sum_{k=0}^{\infty} a_k < \infty.$$

Prove that, for $3^m \le n < 3^{m+1}$, the polynomial

$$t_n(x) = s_{3^m}(f,x) = \sum_{k=0}^{m} a_k \cos 3^k x$$

(which is a partial Fourier series to f) is the b.a. to f from $\mathcal{T}_n$ and find the value of $E_n(f)$.

6.3. Use the previous result to prove the *lethargy theorem*: For any sequence $\varepsilon_n \searrow 0$, there exists an $f \in C(\mathbb{T})$ such that $E_n(f) \ge \varepsilon_n$ all n. (*Hint.* Set $a_k := \varepsilon_{k-1} - \varepsilon_k$.)

6.4. Prove de La Valle-Poussin theorem: If $p \in \mathcal{P}_n$ is a polynomial such that

$$f(t_i) - p(t_i) = (-1)^i \varepsilon_i, \qquad \operatorname{sgn} \varepsilon_i = \text{const},$$

at $n+2$ consecutive points, then $E_n(f) \ge \min_i |\varepsilon_i|$.

6.5. Let $f \in C[a,b]$, and let $p, q \in \mathcal{P}_{n+1}$ be alg. polinomials of degree $n+1$ determined by the condition

$$p(t_i) = f(t_i), \quad q(t_i) = (-1)^i, \qquad \text{where} \quad a \le t_0 < \cdots < t_{n+1} \le b.$$

Prove that the polynomial $r := p - \lambda q$, where λ is chosen so that r belongs to $\mathcal{P}_n$, is the best approximation to f on (t_i) and find the error.

Remark. This is a practical method for constructing the b.a. on the set of $n+2$ points.