Part III - Lent Term 2002

Approximation Theory – Exercise Sheet 2

7.1. Prove that Lemma 7.1 is equivalent to the inequality

$$\sup\{|p_n(\xi)| : \|p_n\| = 1\} = |T_n(\xi)|, \quad |\xi| > 1.$$

Derive from here that

$$|p_n(\xi)| \le |T_n(\xi)|\,\|p_n\| \quad \forall p_n \in \mathcal{P}_n, \quad |\xi| > 1.$$

7.2. Prove that the Chebyshev polynomials are orthogonal with the weight $(1-x^2)^{-1/2}$,

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = 0, \quad \text{if } m \neq n,$$

hence the sequence $(T_k)_{k=0}^n$ forms a basis of $\mathcal{P}_n$.

7.3. Prove that, for any *even* trig. pol. $t_n(\theta) = \sum_{k=0}^n a_k \cos k\theta$ of degree n, the function

$$p_n(x) := t_n(\theta), \quad x = \cos\theta,$$

is an algebraic polynomial of degree n on $[-1,1]$, and vica versa.

7.4. Derive the first (algebraic) Weierstrass theorem from the second (trigonometric) one.

Hint. Given $f \in C[-1,1]$ define a 2π-periodic *even* function $g(\theta) := f(\cos\theta)$, find an *appropriate* trig. approximation $t_n(\theta)$ and make the back substitution.

8.1. *Finite differences.*

For $f \in C(\mathbb{T})$ and $x \in \mathbb{T}$, the finite difference of order k with the step h is defined as

$$\Delta_h^k(f,x) = \Delta_h^1[\Delta_h^{k-1}(f,x)], \qquad \Delta_h^1 f(x) = f(x+h) - f(x).$$

Prove the following identities:

1) $\Delta_h^k(f,x) = \sum_{i=0}^k (-1)^{k-i}\binom{k}{i} f(x+ih)$
2) $\Delta_{nh}^k(f,x) = \sum_{i_1=0}^{n-1}\cdots\sum_{i_k=0}^{n-1}\Delta_h^k(f, x+i_1h+\cdots+i_kh)$,
3) $\Delta_h^k(f,x) = \int_0^h du_1 \int_0^h du_2 \cdots \int_0^h f^{(k)}(x+u_1+\cdots+u_k)du_k$.

Hint. For $k=1$ they are straightforward, for $k>1$ use induction.

8.2. *Moduli of smoothness.*

The k-th modulus of smoothness of $f \in C(\mathbb{T})$ is defined by

$$\omega_k(f,t) = \sup_{0<h<t}\sup_{x\in\mathbb{T}}|\Delta_h^k(f,x)|$$

Making use of [8.1] prove that

(a) $\omega_k(f,t)$ is a contiouous, non-negative and non-decreasing function of t;

(b) $\omega_k(f,nt) \le n^k\,\omega_k(f,t), \quad n \in \mathbb{N}$,

(c) $\omega_k(f,t) \le t^k\,\|f^{(k)}\|$

(d) $\omega_k(f,\lambda t) \le (\lambda+1)^k\,\omega_k(f,t), \quad \lambda \in \mathbb{R}$

8.3. Prove that the Jackson operator admits the estimate in terms of the second modulus of smoothness

$$\|f - L_n(f)\| \le c_1 \omega_2(f, \tfrac{1}{n}) \le c_2 n^{-2} \|f''\|.$$

8.4. For the Fejer operator σ_n, find an estimate

$$\|f - \sigma_n(f)\| \le c\omega(f, \delta_n)$$

with some $\delta_n \to 0$ as $n \to \infty$, the faster, the better.

9.1. Prove the Bernstein inequality for algebraic polynomials

$$|p_n'(x)| \le \frac{n}{\sqrt{1-x^2}} \|p_n\|_\infty \quad \forall p_n \in \mathcal{P}_n[-1,1].$$

9.2. Complete the proof of Theorem 9.4.

10.1. Verify that, for $f(x) = x \ln x$ where $x \in [0,1]$, we have

$$\omega(f,\delta) = \delta \ln \tfrac{1}{\delta}, \qquad \omega_2(f,\delta) = |\Delta_\delta^2 f(0)| = M\delta.$$

10.2. Prove the inverse estimate (10.1)

10.3. Prove the estimate (10.4). Show in between that

$$E_n(f) \le cn^{-1} E_{n-1}(f').$$

Hint. Apply the method used in the trigonometric case.

11.1. Prove Markov's interlacing property for the case when $p \in \mathcal{P}_n$ and $q \in \mathcal{P}_{n-1}$.

11.2. Use Markov's interlacing property to derive the following monotonicity property of zeros of derivative of an algebraic polynomial.

Let $p(x) = \prod_{i=1}^n (x - t_i)$ and let η be a zero of p'. Then η, as a function of t_j, is strictly increasing in the domain $t_{j-1} < t_j < t_{j+1}$.

11.3. Prove the previous statement directly by finding an expression for $\frac{\partial \eta}{\partial t_j}$ and showing that $\frac{\partial \eta}{\partial t_j} > 0$. *Hint.* Differentiate the identity $0 \equiv \frac{p'(\eta)}{p(\eta)}$.

11.4*. Prove the following generalization of Markov's interlacing property to an arbitrary Chebyshev (Haar) system.

Let $f, g \in C^1[a,b]$ be such that any linear combination $\alpha f' + \beta g'$ has no more than $n-1$ zeros. If both f and g have n zeros and these zeros interlace, then zeros of f' and g' interlace too.

Hint. For each $t \in [a,b]$ consider the function

$$h_t(x) := f(x)g'(t) - g(x)f'(t), \quad \text{(so that } h_t'(x) = 0 \text{ at } x = t)$$

Then counting zeros of h_t prove that $h_t(t) \ne 0$ on $[a,b]$. Taking $h_t(t) > 0$, say, and using analysis similar to that given at the end of the proof of Lemma 11.3 derive the conclusion.

12.1. Prove the following generalization of Duffin-Schaeffer theorem.

(Bojanov, Nikolov [1997]) *Let $p = \sum_{k=0}^n a_k T_k$, where $a_k \ge 0$. If $q \in \mathcal{P}_n$ satisfies*

$$|q(\tau_i)| \le |p(\tau_i)| \quad \textit{at the points of local extrema of } p$$

then, for all k,

$$\|q^{(k)}\| \le \|p^{(k)}\| \quad \textit{on} \quad [-1,1].$$

Hint. Prove the analogues of (12.2)-(12.3) for such a p, and apply Theorem 12.1.

12.1. Prove the following variation on the theme.

Let $p(x) = \prod_{i=1}^{n}(x - t_i)$. *If* $q \in \mathcal{P}_n$ *satisfies*

$$|q(x)| \leq |p(x)| \quad \textit{at the points of local extrema of } p,$$

then

$$|q'(x)| \leq |p(x)| \quad \textit{at the points of local extrema of } p'.$$

Hint. Use Lemma 11.7, definition of the Chebyshev sets, and Markov interlacing property.