Part III - Lent Term 2002

Approximation Theory – Exercise Sheet 3

13.1. Prove the scaled estimate (13.4). Find the expression for zeros of T^*_{n+1} on $[a,b]$.

13.2. Making certain changes in formula (13.5) prove that

$$f^{(k)}(x) = p_n^{(k)}(x) + \frac{1}{(n-k)!}\int_a^b (x-\theta)_+^{n-k} f^{(n+1)}(\theta)\,d\theta,$$

hence derive (2) of Lemma 13.2.

13.3. Departing from (13.7), find optimal Δ_* which minimizes the error of Lagrange approximation of the n-th derivative. Comment on uniqueness.

13.4. To get a better feeling of the Zolotarev sets:

Given $\Delta = (t_i)_{i=0}^n$, determine $\mathcal{Z}_\Delta^{(k)}$ for $k=0$, $k=n+1$ and $k=n$.

13.5.* Denote by $|I|$ the length of the interval I. Prove that, for any $\Delta = (t_i)_{i=0}^n$,

$$|\mathcal{Z}_\Delta^{(k)}| := \textstyle\sum_j |\mathcal{Z}_j^{(k)}| = (1 - \frac{k}{n+1})(t_n - t_0).$$

This shows that, for small $k = \mathcal{O}(1)$, the pointwise estimate (13.7) holds almost on the entire $[a,b]$.

Hint. Not very difficult: $|\mathcal{Z}_\Delta^{(k)}| = \sum_{j=1}^{n-k}(\xi_j - \eta_j) = \sum_{j=1}^{n-k}\xi_j - \sum_{j=1}^{n-k}\eta_j$. Try to identify what is $\sum_{j=1}^{n-k}\xi_j$ (or $\frac{1}{n-k}\sum_{j=1}^{n-k}\xi_j$, to be even more precise).

14.1. Let $(\ell_i)_{i=0}^n$ be the sequence of Lagrange fundamental polynomials with respect to a sequence of points $(t_i)_{i=0}^n$. Find a set of functionals (a_i) dual to (ℓ_i).

14.2. Prove that each $s \in \mathcal{S}_{k,m}(\Delta_*)$ can be also written in the form

$$s(x) = q_{k-1}(x) + \sum_{i=1}^N \sum_{j=1}^{m_i} b_{ij}(\tau_i - x)_+^{k-j}\,.$$

(We use $(\tau_i - x)_+^r$ instead of $(x-\tau_i)_+^r$ in (14.1).)

14.3. Consider the space $\mathcal{S} = \mathcal{S}_{2,1}(\Delta_*)$ (of linear splines of deficiency 1) on the knot sequence

$$0 = \tau_0 < 1 = \tau_1 < 2 = \tau_2 < 3 = \tau_3 < 4 = \tau_4$$

Find the representaion of the form (14.1) for $s_i \in \mathcal{S}$, where s_i is given by

$$s_i(\tau_j) = \delta_{ij}, \quad j = 0..4, \quad i = 2,3.$$

14.4. Prove that the Lagrange–Hermite interpolating polynomial exists and is unique.

15.1. We will write $N_i(x) = N(x; t_i, \ldots t_{i+k})$ to make the dependence on knots explicit.

Using the recurrence relation (15.7), prove that, for the so-called Bernstein knots

$$\Delta = \{t_1 = \cdots t_k = 0 < 1 = t_{k+1} = \cdots = t_{2k}\},$$

the corresponding B-splines are

$$N_{i,k}(x) := N(x; \underbrace{0\ldots0}_{k+1-i}, \underbrace{1\ldots1}_{i}) = \binom{k}{i} x^{i-1}(1-x)^{k-i}, \quad i = 1,\ldots,k$$

i.e., $(N_{i,k})$ are the Bernstein basis polynomials (used in the proof of Korovkin theorem).

15.2 Let $t_{k-i} = \cos\frac{\pi i}{k}$, $i = 0, ..., k$, i.e., t_{k-i} are points of equioscillation of the Chebyshev polynomial T_k. Prove that the spline $M(x) := M(x; t_0, ..., t_k)$ is the so-called *perfect* spline, i.e.,

$$|M^{(k-1)}(x)| = \mathrm{const}$$

Hint. From the formula (15.2) find the value $c_\nu := M_i^{(k-1)}(x)$, $x \in [t_\nu, t_{\nu+1})$.

16.1. More about B-splines. Let (t_i) be the integer knot sequence (t_i), i.e., $t_i = i$. Prove that

$$N_i^{(k-1)}(t) = \mathrm{const}\,(-1)^\nu \binom{k-1}{\nu}, \quad t \in (t_{i+\nu}, t_{i+\nu+1})$$

Of course, you need to take only one particular i, say $i = 0$, since the corresponding B-spline sequence (N_i) is *shift invariant*, i.e., $N_i(t) = N_{i-1}(t-1)$.

16.2. Denote by $t_i^{(r)}$ the elementary symmetric f-ns of $k-1$ variables $t_{i+1}, .., t_{i+k-1}$ of degree r,

$$\begin{array}{lcl}
t_i^{(0)} & = & 1, \\
t_i^{(1)} & = & t_{i+1} + \cdots + t_{i+k-1}, \\
t_i^{(2)} & = & t_{i+1}t_{i+2} + t_{i+1}t_{i+3} + \cdots t_{i+k-2}t_{i+k-1}, \\
\multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
t_i^{(k-1)} & = & t_{i+1}t_{i+2}\cdots t_{i+k-1}.
\end{array}$$

Derive from the Marsden identity that monomials $(\cdot)^r$ have the representation

$$t^r = \textstyle\sum_{i=1}^n \binom{k-1}{r}^{-1} t_i^{(r)} N_i(t).$$

17.1. For some functions f, g and for some $x \in \mathbb{R}$, define the functional

$$\lambda(f, g, x) := \sum_{\nu=1}^{k} (-1)^{k-\nu} g^{(\nu-1)}(x) f^{(k-\nu)}(x),$$

whenever the right-hand side exists. If we take $g(x) = \psi_i(x)$, then we obtain the de Boor–Fix functional $\lambda_i(f, x) := \lambda(\psi_i, f, x)$ (see Definition 17.2).

Prove that, for polynomials $p, q \in \mathcal{P}_{k-1}$, the functional $\lambda(p, q, x)$ is independent of x, i.e.,

$$\lambda(p, q, x) \equiv \mathrm{const}(p, q).$$

Hint. Integrate $0 \equiv \int_x^y q^{(k)}(t)p(t)\,dt$.

17.2. A direct construction of the functionals (λ_i) dual to the B-spline basis (N_i) may proceed as follows.

Given $\Delta = (t_i)$, let h_i be a function such that $h_i \in C^k[a, b]$ and

$$h_i(t_j) = 0, \quad j \le i + k - 1; \qquad h_i(t_j) = \psi_i(t_j), \quad j \ge i + 1. \tag{1}$$

Then the functional $\mu_i : C[a, b] \to \mathbb{R}$ given by the rule

$$\mu_i(f) := \textstyle\int_a^b h_i^{(k)}(t) f(t)\,dt \tag{2}$$

satisfies $\mu_i(N_j) = \delta_{ij}$.

Hint. Use the formula $h[t_j...t_{j+k}] = \frac{1}{k!}\int_a^b M_j(t) h^{(k)}(t)\,dt$.

17.3. The simplest way to construct a function h_i satisfying (1) is to set

$$h_i(x) \equiv 0, \quad x < \eta_i; \qquad h_i(x) = \psi_i(x), \quad x > \xi_i, \qquad \eta_i, \xi_i \in (t_i, t_{i+k}), \tag{3}$$

and then to blend smoothly the values in between by posing the conditions

$$h_i^{(\nu-1)}(\eta_i) = 0, \quad h_i^{(\nu-1)}(\xi_i) = \psi_i^{(\nu-1)}(\xi_i), \quad \nu = 1, ..., k, \tag{4}$$

(and adding the conditions $h_i(x) = 0$ at $x = t_{i+1}, ..., t_{i+k-1}$, when necessary).

Let $\mu_i(f)$ be given by (2)-(4). Show that

$$\mu_i(f) = \lambda_i(f, \xi_i) \quad \text{for} \quad f \in \mathcal{S}_k(\Delta),$$

i.e., if f belongs to $\mathcal{S}_k(\Delta)$, then $\mu_i(f)$ coincides with the de Boor–Fix functional $\lambda_i(f, \xi_i)$.

(This means that the formulas (2)–(4) provide an extension of $\lambda_i(f)$ from the subspace $\mathcal{S}_k(\Delta)$ of $C(I)$ to the entire space $C(I)$. Notice also that $h_i^{(k)}(x) \equiv 0$ outside $[\eta_i, \xi_i]$, i.e., the functional μ_i has a local support.)

Hint. Start with the case $[\eta_i, \xi_i] \subset [t_\ell, t_{\ell+1}]$ for some ℓ, then turn to the case when the interval (η_i, ξ_i) contains several points t_j of the knot sequence.

17.4. The simplest way to construct a function h_i satisfying (4) is to set

$$h_i(x) := g_i(x)\psi_i(x),$$

where $g_i \in C^k[\eta_i, \xi_i]$ is a function such that

$$g_i(x) = \begin{cases} 0, & k\text{-fold at } x = \eta_i; \\ 1, & k\text{-fold at } x = \xi_i. \end{cases} \tag{5}$$

Take $[\eta_i, \xi_i] = [t_i, t_{i+k}]$. Then

$$\mu_i(f) = \int_{t_i}^{t_{i+k}} [g_i(t)\psi_i(t)]^{(k)} f(t)\, dt$$

and the Hölder inequality $|\int fg| \le \|f\|_\infty \|g\|_1$ implies

$$|\mu_i(f)| \le c\, \|f\|_{C[t_i, t_{i+k}]}, \quad c := \|(g_i\psi_i)^{(k)}\|_{L_1[t_i, t_{i+k}]}.$$

Choosing an appropriate g_i satisfying (5), show that $c \le c_k$, i.e., that μ_i is a Hahn–Banach type extension of λ_i.

18.1. On considering the case $k = 1$, prove that a Hahn-Banach extension is not unique in the sense that extension of $\lambda_i(s) = \lambda_i(s, \xi_i)$ may depend on ξ_i.

18.2. Prove that the linear quasi-interpolant

$$Q(f, x) = \sum_{i=1}^{n} f(t_{i+1}) N_i(x)$$

(which is now an interpolant) satisfies the estimate

$$\|f - Qf\| \le \tfrac{1}{8}|t|^2\, \|f''\|$$