Part III - Lent Term 2002

Approximation Theory – Exercise Sheet 4

19.1. (*Exam question 2001*)

1) Consider the following case of parabolic spline (actually, polynomial) interpolation:

$$I = [0, 1], \quad k = 3, \quad \boldsymbol{x} = (0, \tfrac{1}{2}, 1), \quad \Delta = (0, 0, 0, 1, 1, 1).$$

The B-splines on the Bernstein knots are the fundamental Bernstein polynomials

$$N_1(x) = (1 - x)^2, \quad N_2(x) = 2x(1 - x), \quad N_3(x) = x^2.$$

Prove that $\|A_{\boldsymbol{x}}^{-1}\|$ provides the bound $\|P_{\boldsymbol{x}}\| \le 3$.

2) Prove directly that the exact value of the norm (of parabolic interpolation on $\boldsymbol{x}$) is $\|P_{\boldsymbol{x}}\| = \frac{7}{4}$, hence the bound (19.3) is not sharp.

19.2 Prove the following version of the Schoenberg-Whitney theorem (which refers to the basis of truncated powers).

Let $\mathcal{S}_k(\Delta_*)$ be the space of the spline function of degree $k - 1$ with the knot-sequence

$$\tau_1 < \tau_2 < \cdots < \tau_{n-k}$$

and let $\boldsymbol{x} = (x_1 < \cdots < x_n)$. Then interpolation problem

$$s(x_i) = y_i, \quad s \in \mathcal{S}_k(\Delta_*)$$

is uniquely solvable for any $\boldsymbol{y}$ if and only if

$$x_i < \tau_i < x_{i+k} \tag{1}$$

Hint. Prove that determinant of the matrix

$$B_{\boldsymbol{x}} = \begin{bmatrix} 1 & x_1 & \dots & x_1^{k-1} & (x_1 - \tau_1)_+^{k-1} & \dots & (x_1 - \tau_{n-k})_+^{k-1} \\ \multicolumn{7}{c}{\dotfill} \\ 1 & x_{n-k} & \dots & x_{n-k}^{k-1} & (x_{n-k} - \tau_1)_+^{k-1} & \dots & (x_{n-k} - \tau_{n-k})_+^{k-1} \\ \multicolumn{7}{c}{\dotfill} \\ 1 & x_n & \dots & x_n^{k-1} & (x_n - \tau_1)_+^{k-1} & \dots & (x_n - \tau_{n-k})_+^{k-1} \end{bmatrix}$$

does not vanish iff (1) is fulfilled.

For the 'if' part, using induction on n, prove that $\det B_{\boldsymbol{x}}$ as a function of $\tau = \tau_{n-k}$ is a B-spline with knots $(x_{n-k}, ..., x_n)$, i.e.,

$$\det B_{\boldsymbol{x}}(\tau) = cN(\tau; x_{n-k}, ..., x_n), \quad c \ne 0.$$

(Expand the determinant by elements of the last columns.)

20.1. Let $\omega(x) = (x - t_0)...(x - t_k)$, $x_i \ne x_j$. Prove that

$$\frac{1}{\omega(x)} = [t_0, ..., t_k] \, \frac{1}{(x - \cdot)}$$

20.2. Assuming the continuity of $f[t_0...t_k]$ with respect to t_j for sufficiently smooth function, prove that, for

$$(t_0, ..., t_k) = (\underbrace{\tau_r, .., \tau_r}_{m_r}, ..., \underbrace{\tau_s, .., \tau_s}_{m_s}),$$

the coefficients $a_{\mu\nu}$ in the formula (20.3) satisfy

$$\frac{1}{\omega(x)} = \sum_{\mu=r}^{s} \sum_{\nu=1}^{m_\mu} a_{\mu\nu} \frac{(\nu-1)!}{(x-\tau_\mu)^\nu}.$$

In other words, they provide the decomposition of $\frac{1}{\omega(x)}$ into *partial fractions*.

20.3 Let $s = \sum_i a_i N_i$ and let $t_i^* := (t_{i+1} + ... + t_{i+k-1})/(k-1)$. Since $x = \sum_i t_i^* N_i(x)$, we can represent the spline *curve* $(x, s(x))$ as a vector-valued spline $\sum_i P_i N_i$ with coefficients $P_i = (t_i^*, a_i)$ called its *control* points. The broken line Γ_s with the vertices (t_i^*, a_i) is called the *control polygon*.

Verify that, by the knot insertion, the points t_i satisfy

$$\widehat{t_i^*} = \gamma_i t_i^* + (1-\gamma_i) t_{i-1}^*,$$

i.e., the same relation as for the refined B-spline coefficients.

Therefore (compare with the proof of Lemma 20.7), the refined control polygon $\widehat{\Gamma}_s$ interpolates, at its breakpoints, the original one. If $\widehat{t}_j$ is inserted $(k-1)$ times, then $\widehat{\Gamma}_s(\widehat{t}_j) = s(\widehat{t}_j)$.

21.1. (*Exam question 2001*)

1) Consider the following case of parabolic spline (actually, polynomial) interpolation:

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The B-splines on the Bernstein knots are the fundamental Bernstein polynomials

$$N_1(x) = (1-x)^2, \quad N_2(x) = 2x(1-x), \quad N_3(x) = x^2.$$

Prove that $\|A_{\boldsymbol{x}}^{-1}\|$ provides the bound $\|P_{\boldsymbol{x}}\| \le 3$.

2) Prove directly that the exact value of the norm (of parabolic interpolation on $\boldsymbol{x}$) is $\|P_{\boldsymbol{x}}\| = \frac{7}{4}$, hence the bound (21.1) is not sharp.

21.2. Prove that the matrix

$$A = \begin{bmatrix} 1 & \cdots & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

is totally positive.

21.3. Let A be invertible and totally positive. If a, b are the vectors such that $Aa = b$ and $(-1)^i b_i \ge \gamma > 0$, then

$$\|A^{-1}\|_\infty \le \tfrac{1}{\gamma} \|a\|_\infty.$$

21.4. Will the statements of Lemma 20.7 and of Theorem 20.8 remain true for the number S^+ of weak sign changes instead of S^-?

22.1. Using the formula (of Exercise 4.4a) and some details of the proof of Lemma 22.3

$$|\lambda(f)| = \|\lambda\| \operatorname{dist}(f, \ker \lambda), \quad \forall f \in \mathbb{X}, \quad \forall \lambda \in \mathbb{X}^*, \tag{1}$$

prove that

$$\|\lambda_i\| = \frac{1}{\operatorname{dist}(N_i, \operatorname{span}_{j\ne i} N_j)}. \tag{2}$$

Proof of (1). For any $g \in \ker\lambda$, we obtain $|\lambda(f)| = |\lambda(f-g)| \le \|\lambda\|\,\|f-g\|$, hence

$$|\lambda(f)| \le \|\lambda\| \operatorname{dist}(f, \ker\lambda).$$

For the opposite inequality, let (f_i) satisfy

$$|\lambda(f_i)| \ge (1-\varepsilon_i)\|\lambda\|\,\|f_i\|, \quad \lambda(f_i) = \lambda(f).$$

Then $f_i - f \in \ker\lambda$ and

$$\lambda(f) \ge (1-\varepsilon_i)\|\lambda\|\,\|f + (f_i - f)\| \ge (1-\varepsilon_i)\|\lambda\| \operatorname{dist}(f, \ker\lambda).$$

22.2. Derive (2) directly from the definition of λ.

22.3. *B-spline basis condition number.* The condition number of a basis V of an n-dimensional normed linear space $\mathbb{X}$ is the value

$$\kappa(V) := \sup_a \frac{\|a\|_\infty}{\|\sum_i a_i v_i\|} \cdot \sup_b \frac{\|\sum_i b_i v_i\|}{\|b\|_\infty}.$$

It measures the extent to which relative changes in the coordinates a of an element $Va := \sum_i a_i v_i$ may be close to the relative change in Va itself.

Alternatively, $\kappa(V) := \frac{c_2}{c_1}$ where c_1, c_2 are the *best* constants in the inequalities

$$c_1\,\|a\|_\infty \le \|\textstyle\sum_i a_i v_i\| \le c_2\,\|a\|_\infty.$$

Prove that for the B-spline basis $B = (N_i)$

$$\kappa(B) = \max_i \|\lambda_i\|.$$

Hint. Show that $c_2 = 1$.

23.1. Prove that if $G = [(u_i, u_j)]$ then the elements of the Gram inverse are

$$(G^{-1})_{ij} = (u_i^*, u_j^*).$$

23.2. Prove that, for any basis (u_i) of $\mathcal{U}_n$, the Gram matrix $G = [(u_i, u_j)]$ is positive definite, i.e., $(Gx, x)_{\ell_2} > 0$ for any nonzero vector $x \in \mathbb{R}^n$.

Prove the converse: for any positive definite matrix $G \in \mathbb{R}^{n\times n}$, there exists a basis (u_i) of $\mathcal{U}_n$ such that $G = [(u_i, u_j)]$.

23.3. (*Smooth interpolation.*) Let $\mathbb{X}$ be a Hilbert space (a complete inner product space), let $f \in \mathbb{X}$, and let the available information about f consists of interpolating data $\mu_i(f)$ with respect to some sequence of functionals (μ_i). We seek a (quasi-) approximation u to f that satisfies the same interpolating conditions, but among many candidates we choose the "smoothest" one in the sense that it has the least norm

$$\widehat{u} = \arg\min\{\|u\| : \mu_i(u) = \mu_i(f)\}.$$

By Riesz theorem, any linear functional on the Hilbert space $\mathbb{X}$ can be represented as the scalar product,

$$\mu_i(f) = (f, u_i), \quad u_i \in \mathbb{X}.$$

Prove that $\widehat{u}$ coincides with u^*, the best approximation to f from $\mathcal{U}_n := \operatorname{span}(u_i)$.

23.4. (*Optimal recovery.*) Condiser the problem:

$$\text{given functionals } (\mu_i)_{i=1}^n \text{ and } \lambda, \text{ and the values } \mu_i(f) = y_i \quad \to \quad \text{evaluate } \lambda(f)$$

(e.g., a quadrature rule $\lambda(f) = \int f$ or a formula for numerical differentiation $\lambda(f) = f'(x_0)$). We expect that $\lambda(f)$ has a finite value, so we need to put some restriction on f, say $\|f\| \le M$. Set

$$D := \{f \in \mathbb{X} : \mu_i(f) = y_i, \|f\| \le M\}$$

Then the values of the functional λ on D lie inside a certain finite interval

$$a \le \inf_{f \in D} \lambda(f) \le \sup_{f \in D} \lambda(f) \le b.$$

Clearly, the optimal approximation (or, *recovery*) of λ on D is the value $\frac{a+b}{2}$.

Prove: the smoothest interpolant $\widehat{u}$ to (y_i) gives the optimal recovery to any λ, i.e.,

$$\lambda(\widehat{u}) = \tfrac{a+b}{2}.$$

23.5*. (*Inverse of a band matrix.*) An exponential decay of the Gramian inverse (or, equivalently, of the norms of dual elements) was proved originally for the B-spline basis. Its generalization to the finitely supported and well-conditioned bases (as given in Proposition 23.8) appeared by analyzing which properties of the spline function are actually involved in the proof. Subsequently, Demko eliminated even the bases and reduced the statement to the pure fact about band matrices.

Demko theorem. Let A be a band matrix, that is bounded and has a bounded inverse, i.e.,

$$\begin{aligned} &1') \quad a_{ij} = 0 \quad \text{if} \quad |i-j| \ge k. \\ &2') \quad \|A\|_2 = c_1, \quad \|A^{-1}\|_2 = c_2. \end{aligned}$$

Then the elements of $A^{-1} = (b_{ij})$ decays exponentially away from the diagonal, i.e.,

$$|b_{ij}| \le c\,\gamma^{|i-j|}, \quad \gamma < 1,$$

where c and γ depend on c_1, c_2 and k. (For you to prove.)

Remark. This is how the Gramian $G = (u_i, u_j)$ of the $\mathcal{S}_{k,d}$-basis fits into Demko theorem:

1) Finiteness of support (23.2) provides exactly (1′)
2) The basis condition (23.3) is the inequality $d^{-2}\|x\|^2 \le (Gx, x) \le \|x\|^2$ which implies

$$d^{-2} \le \lambda_{\min}(G) = \|G^{-1}\|_2^{-1}, \quad \|G\|_2 = \lambda_{\max}(G) \le 1.$$

Hint. Try to obtain analogues of (23.6)-(23.7) by considering the vectors

$$b^{(m)} = (0, ..., 0, b_m, b_{m+1}, ..., b_n)$$

and by showing that, for some $r = r(k)$, the vectors $Ab^{(m)}$ and $Ab^{(m+r)}$ have non-overlapping supports. (Support of a vector is a sequence of the coordinates of its non-zero components).

You may also find useful the inequalities

$$\|Ax\| \le \|A\|\,\|x\|\,, \qquad \|Ax\| \ge \|A^{-1}\|^{-1}\,\|x\|\,.$$

24.1. A matrix $A \in \mathbb{R}^{n\times n}$ is called *strictly diagonally dominant* by rows if

$$|a_{ii}| - \sum_{j\neq i} |a_{ij}| \geq \gamma > 0 \quad i = 1, ..., n. \tag{1}$$

Example: $A = \begin{bmatrix} 4 & 1 & & \\ 1 & 4 & 1 & \\ & 1 & 4 & \end{bmatrix}$. Prove that, if (1) holds, then $\|A^{-1}\|_\infty \leq 1/\gamma$.

Hint. Use equality $\|A^{-1}\|_\infty = \sup_x \frac{\|x\|_\infty}{\|Ax\|_\infty}$.

24.2. 1) For $k = 2$ and for any Δ, compute the entries of $G = (M_i, N_j)$.

Hint. This Gramian is tri-diagonal and 'shift-invariant' by rows in the sense that the next row is obtained from the previous one by the shift $t_i \to t_{i+1}$. Hence only three elements $g_{i,i-1}$, $g_{i,i}$ and $g_{i,i+1}$ need to be computed. Notice that the sum of elements in any row must be equal to one, since

$$\sum_{j=1}^{n} g_{ij} = \sum_{j=1}^{n} (M_i, N_j) = (M_i, \sum_{j=1}^{n} N_j) = (M_i, 1) = 1.$$

2) Prove the theorem by Ciesielski [1963]: For $k = 2$ and for *any* Δ,

$$\|P_{\mathcal{S}}\|_\infty \leq 3.$$

24.3. Find the *best* (i.e., the least possible) constant c_n in the Markov inequality for L_2- and L_∞-norms,

$$\|q_n\|_\infty \leq c_n \|q_n\|_2, \quad q_n \in \mathcal{P}_n[-1, 1].$$

Hint. You may use the fact that the Legendre orthonormal polynomials p_i such that

$$\textstyle\int_{-1}^{1} p_i(t) p_j(t) = \delta_{ij},$$

satisfy

$$\|p_k\|_\infty = |p_k(1)| = \sqrt{k + \tfrac{1}{2}}.$$