New upper bound for the B-spline basis condition number

K. Scherer

Institut für Angewandte Mathematik, Universität Bonn, 53115 Bonn, Germany

and

A. Yu. Shadrin [1]

Computing Center, Siberian Branch, Russian Academy of Sciences,

630090 Novosibirsk, Russia

Abstract

For the p-norm condition number $\kappa_{k,p}$ of the B-spline basis of order k we prove the upper estimate

$$\kappa_{k,p} < k^{1/2}\, 4^k.$$

This improves de Boor's estimate $\kappa_{k,p} < k\, 9^k$, and stands closer to his conjecture that $\kappa_{k,p} \sim 2^k$.

[1] Supported by a grant from the Alexander von Humboldt – Stiftung

1. Introduction

Let $\{\hat{N}_j\}$ be the B-spline basis of order k on a feasible knot sequence $t = \{t_j\}$, $t_j < t_{j+k}$, normalized with respect to the L_p-norm ($1 \le p \le \infty$), i.e.

$$\hat{N}_j(x) = (k/(t_{j+k} - t_j))^{1/p} N_j(x),$$

where $\{N_j\}$ are the B-splines forming a partition of unity.

The condition number of $\{\hat{N}_j\}$ is defined as

$$\begin{aligned} \kappa_{k,p,t} &:= \sup_{(b)} \frac{\|(b)\|_{l_p}}{\|\sum b_j \hat{N}_j\|_{L_p}} \sup_{(b)} \frac{\|\sum b_j \hat{N}_j\|_{L_p}}{\|(b)\|_{l_p}} \\ &= \sup_{(b)} \frac{\|(b)\|_{l_p}}{\|\sum b_j \hat{N}_j\|_{L_p}}, \end{aligned}$$

where the L_p-norm is taken with respect to the smallest interval containing the knots $\{t_i\}$.

Set

$$\kappa_{k,p} := \sup_t \kappa_{k,p,t},$$

defining thus the worst B-spline condition number.

C.de Boor [1] proved that

$$\kappa_{k,p} \le 2k\, 9^{k-1}, \tag{1.1}$$

and conjectured that

$$\kappa_{k,p} \sim 2^k.$$

This conjecture was supported by the lower bound of T.Lyche [4]

$$\kappa_{k,\infty} \ge (k-1)k^{-1}\, 2^{k-3/2}$$

and by recent numerical computations [2]. However, the poor upper estimate (1.1) remained unchanged.

Here we prove

Theorem 1.

$$\kappa_{k,p} < k^{1/2}\, 4^k.$$

2. Preliminaries

Following de Boor [1], we will introduce here some related constants which majorize $\kappa_{k,p}$ for all $p \in [1, \infty]$. More details on the problems relevant to $\kappa_{k,p}$ can be found in [1],[2],[5].

Recall that

$$N_j(t) := (t_{j+k} - t_j)\, [t_j, t_{j+1}, \ldots, t_{j+k}]\, (\cdot - t)_+^{k-1},$$

so that

$$\operatorname{supp} N_j = (t_j, t_{j+k}), \quad N_j \ge 0, \quad \sum N_j = 1.$$

Lemma A. *Let H_i be the class of functions $h_i \in L_\infty$ such that*

1) $\operatorname{supp} h_i \subset [t_i, t_{i+k}]$

2) $\int h_i N_j = \delta_{ij}$.

and let

$$D_k = \sup_t \sup_i \inf_{h_i \in H_i} \{(t_{i+k} - t_i)\|h_i\|_\infty\}.$$

Then

$$\kappa_{k,p} \le D_k.$$

Set

$$\psi_i(x) = \frac{1}{(k-1)!}\prod_{\nu=1}^{k-1}(x - t_{i+\nu}).$$

The following lemma shows how the functions $h_i \in H_i$ could be constructed.

Lemma B. *Let F_i be the class of functions f_i such that*

$$1) \quad f_i \in W_\infty^k[t_i, t_{i+k}],$$

$$2) \quad f_i = \begin{cases} 0, & k\text{-fold at } t_i, \\ 0 = \psi_i, & \forall t_j \in (t_i, t_{i+k}), \\ \psi_i, & k\text{-fold at } t_{i+k}, \end{cases}$$

and let $F_i^{(k)} = \{f_i^{(k)} : f_i \in F_i\}$. Then

$$F_i^{(k)} \subset H_i.$$

Further, an easy way for obtaining $f_i \in F_i$ is to set $f_i = g_i\psi_i$ with appropriate smoothing function g_i. We formulate it as

Lemma C. *Let G_i be the class of functions g_i such that*

$$1) \quad g_i\psi_i \in W_\infty^k[t_i, t_{i+k}],$$

$$2) \quad g_i\psi_i = \begin{cases} 0, & k\text{-fold at } t_i, \\ \psi_i, & k\text{-fold at } t_{i+k}, \end{cases}$$

and let $G_i^{(k)} = \{(g_i\psi_i)^{(k)} : g_i \in G_i\}$. Then

$$G_i^{(k)} \subset F_i^{(k)}.$$

We summarize Lemmas A-C as follows.

Corollary. *Let*

$$B_k = \sup_t \sup_i \inf_{g_i \in G_i} \left\{(t_{i+k} - t_i)\|(g_i\psi_i)^{(k)}\|_\infty\right\}.$$

Then

$$\kappa_{k,p} \le B_k.$$

Finally, due to the local character of all the statements, we reformulate the final result making the linear transform $[t_i, t_{i+k}] \to [0,1]$, eliminating thereby the reference to the meshes t.

Denote by Π_{k-1} the set of algebraic polynomials ω of degree $k-1$ with all their zeros lying in $[0,1]$ and higher derivative equal 1, i.e. of the form

$$\omega(x) = \frac{1}{(k-1)!}\prod_1^{k-1}(x - t_i), \quad t_i \in [0,1].$$

Lemma D. *For $\omega \in \Pi_{k-1}$ given, let G_ω be the class of functions g such that*

$$1) \quad g\omega \in W_\infty^k[0,1]$$

$$2) \quad g\omega = \begin{cases} 0, & k\text{-fold at } 0, \\ \omega, & k\text{-fold at } 1, \end{cases}$$

and let

$$B_k = \sup_{\omega \in \Pi_{k-1}} \inf_{g \in G_\omega} \|(gw)^{(k)}\|_\infty .$$

Then

$$\kappa_{k,p} \le B_k .$$

Remark. Lemmas A and B are taken from [1, p.123] and [1, p.127]. Lemmas C and, respectively, D are somewhat more accurate version of what is given in [1, Eq.(4.1)]. Namely, they show the possibility to choose the smoothing function g *depending* on ω. C.de Boor's estimate of B_k resulted in (1.1) was based on the inequalities

$$\begin{aligned} B_k &\le \inf_{g \in G} \sup_{\omega} \|(g\omega)^{(k)}\| \le \inf_{g \in G} \sum_{m=1}^{k} \binom{k}{m} \|g^{(m)}\| \sup_{\omega} \|\omega^{(k-m)}\| \\ &\le \sum_{m=1}^{k} \binom{k}{m} \|g_*^{(m)}\| \sup_{\omega} \|\omega^{(k-m)}\|, \end{aligned}$$

with some special choice of $g_* \in G := \cap G_\omega$ which is seen to be *independent* of ω. Notice also, that in the latter sum for any choice of $g_* \in G$ the term with $m = k$ is equal at least to 4^{k-1} (see [1, p.132]).

3. Proof of Theorem 1

We will estimate B_k choosing $g \in G_\omega$ depending on ω in the following way. For $\omega \in \Pi_{k-1}$,

$$\omega(x) = \frac{1}{(k-1)!} \prod_{i=1}^{k-1} (x - t_i),$$

set

$$g_\omega(x) = \int_0^x M_\omega(t)\, dt,$$

where M_ω is the B-spline of degree $k-1$ constructed on the mesh

$$0 = t_0 \le t_1 \le \ldots \le t_{k-1} \le t_k = 1,$$

and normalized with respect to the L_1-norm, i.e.,

$$\int_0^1 M_\omega(t)\, dt = 1.$$

Example. In the case of the Bernstein knots, when

$$\omega(x) = c_1\, x^\alpha (x-1)^\beta, \quad c_1 = 1/(k-1)! = 1/(\alpha+\beta)!\, ,$$

we have

$$M_\omega(t) = c_2\, t^\beta (1-t)^\alpha, \quad c_2 = k\binom{k-1}{\alpha}.$$

Next we need two lemmas which proofs will be given below.

LEMMA 1. *For any* $\omega \in \Pi_{k-1}$

$$g_\omega \in G_\omega.$$

LEMMA 2. *For any* $\omega \in \Pi_{k-1}$

$$\|g_\omega^{(m)} \cdot \omega^{(k-m)}\| \le k\binom{k-1}{m-1}, \quad m = 1,\dots,k.$$

Now by Lemmas 1,2, and D,

$$\begin{aligned} \kappa_{k,p} \le B_k \quad &\le \quad \sup_\omega \|(g_\omega \cdot \omega)^{(k)}\| \le \sum_{m=1}^{k} \binom{k}{m} \sup_\omega \|g_\omega^{(m)} \cdot \omega^{(k-m)}\| \\ &\le \quad k \sum_{m=1}^{k} \binom{k}{m}\binom{k-1}{m-1} = k \cdot \frac{1}{2}\binom{2k}{k}, \end{aligned} \tag{3.1}$$

that is

$$\kappa_{k,p} \le \frac{k}{2}\binom{2k}{k}.$$

Finally, with respect to Wallis' inequality

$$4^k/\sqrt{(k+1/2)\pi} < \binom{2k}{k} < 4^k/\sqrt{k\pi}$$

we have

$$\kappa_{k,p} < \frac{1}{2\sqrt{\pi}}\, k^{1/2}\, 4^k < k^{1/2}\, 4^k.$$

4. Proof of Lemma 1

Let us verify that

$$\begin{aligned} &1) \quad g_\omega \cdot \omega \in W_\infty^k[0,1] \\ &2) \quad g_\omega \cdot \omega = \begin{cases} 0, & k\text{-fold at } 0, \\ \omega, & k\text{-fold at } 1, \end{cases} \end{aligned} \tag{4.1}$$

1) This is evidently true, if ω has only simple zeros $\{t_i\}_1^{k-1}$ lying strictly in $(0,1)$, i.e. for

$$0 = t_0 < t_1 < \ldots < t_{k-1} < t_k = 1.$$

In fact, in this case

$$\begin{aligned} &1') \quad M_\omega \in W_\infty^{k-1}[0,1], \\ &2') \quad M_\omega^{(\nu)}(x)\big|_{x=0,1} = 0, \quad \nu = 0,\dots,k-2. \end{aligned}$$

For $g_\omega = \int M_\omega$ this provides

$$1'') \quad g_\omega \in W_\infty^k[0,1]$$

$$2'') \quad g_\omega = \begin{cases} 0, & k\text{-fold at } 0, \\ 1, & k\text{-fold at } 1, \end{cases}$$

which in turn implies (4.1).

2) If ω has a multiple zero

$$\tau_\nu := t_{\mu_\nu} = t_{\mu_\nu+1} = \ldots = t_{\mu_\nu+p_\nu-1},$$

then the smoothness of M_ω and $g_\omega = \int M_\omega$ respectively will drop down at the knot τ_ν to the amount $p_\nu - 1$, but this loss will be compensated in the product $g_\omega \cdot \omega$ by the factor $(x-\tau_\nu)^{p_\nu-1}$ from ω. (See Example in Sect. 3).

5. Proof of Lemma 2

Since $g_\omega^{(m)}(x) := M_\omega^{(m-1)}(x)$, we need to prove the following statement.

Lemma 2′.

$$(5.1) \qquad \sup_{\omega\in\Pi_{k-1}} \frac{t_k - t_0}{k} \|M_\omega^{(m-1)} \cdot \omega^{(k-m)}\| \le \binom{k-1}{m-1}, \quad m = 1,\ldots,k.$$

Remark. If ω has a zero τ_ν of multiplicity p_ν, then $M_\omega^{(k-p_\nu)}$ has a jump at τ_ν. In this case we can define the value $M_\omega^{(k-p_\nu+\mu)}(\tau_\nu)\,\omega^{(p_\nu-1-\mu)}(\tau_\nu)$ as a limit either from the left or from the right. This limit is equal to zero, if $\tau_\nu \in (t_0, t_k)$. Also, this definition justifies the equality

$$(g_\omega \cdot \omega)^{(k)}(x) = \sum_{m=1}^k \binom{k}{m} g_\omega^{(m)}(x)\omega^{(k-m)}(x) := \sum_{m=1}^k \binom{k}{m} M_\omega^{(m-1)}(x)\omega^{(k-m)}(x),$$

which was used in (3.1).

Proof. This will be given by induction on k. Namely, we establish some recurrence relations for the left-hand side of (5.1).

Denote by $M_{j,p}$ the B-spline of degree p such that

$$\operatorname{supp} M_{j,p} = (t_j, t_{j+p+1}), \qquad \int_{t_j}^{t_{j+p+1}} M_{j,p}(x)\,dx = 1,$$

and by $\omega_{j,p}$ the corresponding polynomial of the same degree p

$$\omega_{j,p}(x) = \frac{1}{p!}(x-t_{j+1})\ldots(x-t_{j+p}).$$

whose zeros coincide with the inner knots of $M_{j,p}$.

We establish some recurrence relations for the values

$$(5.2) \qquad \begin{aligned} A_{p,r} &:= \sup \frac{t_{j+p+1}-t_j}{p+1} \|M_{j,p}^{(r)} \omega_{j,p}^{(p-r)}\| \\ B_{p,r} &:= \sup \frac{1}{p+1} \|M_{j,p}^{(r)} \omega_{j,p}^{(p-r-1)}\|. \end{aligned}$$

where sup is taken over all the meshes $t_j \le \ldots \le t_{j+p+1}$.

For this purpose it will be enough to use three B-splines

$$M := M_{0,p}; \quad \begin{cases} M_0 := M_{0,p-1}, \\ M_1 := M_{1,p-1}; \end{cases}$$

and, respectively, three polynomials

$$\omega := \omega_{0,p}; \quad \begin{cases} \omega_0 := \omega_{0,p-1}, \\ \omega_1 := \omega_{1,p-1}; \end{cases}$$

for which, by definition,

$$\omega(x) = \frac{1}{p}\,(x - t_p)\,\omega_0(x) = \frac{1}{p}\,(x - t_1)\,\omega_1(x). \tag{5.3}$$

1) First, we use the following identity [3, Eq.(4.6)] for the B-splines M, M_0, M_1

$$\frac{t_{p+1} - t_0}{p+1}\, M^{(r)}(x) = \left[M_0^{(r-1)}(x) - M_1^{(r-1)}(x) \right],$$

coupled with the equalities for $\omega, \omega_0, \omega_1$ obtained from (5.3) by simple differentiation:

$$\begin{aligned} \omega^{(p-r)}(x) &= p^{-1} \left[(x - t_p)\,\omega_0^{(p-r)}(x) + (p - r)\,\omega_0^{(p-r-1)}(x) \right] \\ &= p^{-1} \left[(x - t_1)\,\omega_1^{(p-r)}(x) + (p - r)\,\omega_1^{(p-r-1)}(x) \right]. \end{aligned}$$

This gives

$$\begin{aligned} &\frac{t_{p+1} - t_0}{p+1}\, M^{(r)}(x)\,\omega^{(p-r)}(x) \\ &= p^{-1} \Big[(x - t_p) M_0^{(r-1)}(x)\,\omega_0^{(p-r)}(x) + (p - r) M_0^{(r-1)}(x)\,\omega_0^{(p-r-1)}(x) \\ &\quad - (x - t_1) M_1^{(r-1)}(x)\,\omega_1^{(p-r)}(x) - (p - r) M_1^{(r-1)}(x)\,\omega_1^{(p-r-1)}(x) \Big]. \end{aligned} \tag{5.4}$$

Now, by definition (5.2), and due to the finiteness of B-splines

$$\begin{aligned} p^{-1} \big| M_0^{(r-1)}(x)\,\omega_0^{(p-r)}(x) \big| &:= p^{-1} \big| M_{0,p-1}^{(r-1)}(x)\,\omega_{0,p-1}^{(p-1-(r-1))}(x) \big| \\ &\le \frac{\chi_{[t_0,t_p]}(x)}{t_p - t_0}\, A_{p-1,r-1}, \end{aligned}$$

with χ_E being the characteristic function of the interval E. Similarly,

$$p^{-1} \big| M_1^{(r-1)}(x)\,\omega_1^{(p-r)}(x) \big| \le \frac{\chi_{[t_1,t_{p+1}]}(x)}{t_{p+1} - t_1}\, A_{p-1,r-1}.$$

Also, by (5.2)

$$p^{-1} \big| M_0^{(r-1)}(x)\,\omega_0^{(p-r-1)}(x) \big| := p^{-1} \big| M_{0,p-1}^{(r-1)}(x)\,\omega_{0,p-1}^{(p-1-(r-1)-1)}(x) \big| \le B_{p-1,r-1},$$

and, respectively,

$$p^{-1} \big| M_1^{(r-1)}(x)\,\omega_1^{(p-1-r)}(x) \big| \le B_{p-1,r-1}.$$

Thus, from (5.4) we derive

$$A_{p,r} \le \|D\|\, A_{p-1,r-1} + 2(p - r)\, B_{p-1,r-1},$$

where

$$D(x) = \frac{|x-t_p|}{t_p-t_0}\,\chi_{[t_0,t_p]}(x) + \frac{|x-t_1|}{t_{p+1}-t_1}\,\chi_{[t_1,t_{p+1}]}(x).$$

Evidently $D(x) \le 1$, hence

$$A_{p,r} \le A_{p-1,r-1} + 2(p-r)\,B_{p-1,r-1}. \tag{5.5}$$

2) Next we find a similar recurrence relation for

$$B_{p,r} := \sup_t \frac{1}{p+1}\,\|M_{j,p}^{(r)}\omega_{j,p}^{(p-r-1)}\|.$$

For this purpose we use another identity [3, Eq.(4.9)] for the B-splines M, M_0, M_1

$$\frac{1}{p+1}\,M^{(r)}(x) = \frac{1}{t_{p+1}-t_0}\,\frac{1}{p-r}\left[(x-t_0)M_0^{(r)}(x) + (t_{p+1}-x)M_1^{(r)}(x)\right],$$

coupled with the previous relations for $\omega, \omega_0, \omega_1$

$$\begin{aligned}
\omega^{(p-r-1)}(x) &= p^{-1}\left[(x-t_p)\,\omega_0^{(p-r-1)}(x) + (p-r-1)\,\omega_0^{(p-r-2)}(x)\right] \\
&= p^{-1}\left[(x-t_1)\,\omega_1^{(p-r-1)}(x) + (p-r-1)\,\omega_1^{(p-r-2)}(x)\right].
\end{aligned}$$

This implies

$$\begin{aligned}
&\frac{1}{p+1}\,M^{(r)}(x)\,\omega^{(p-r-1)}(x) \\
&\quad= \frac{1}{p}\,\frac{1}{t_{p+1}-t_0}\,\frac{1}{p-r}\times\Big[(x-t_0)(x-t_p)M_0^{(r)}(x)\,\omega_0^{(p-r-1)}(x) \\
&\qquad\qquad + (p-r-1)(x-t_0)M_0^{(r)}(x)\,\omega_0^{(p-r-2)}(x) \\
&\qquad\qquad + (t_{p+1}-x)(x-t_1)M_1^{(r)}(x)\,\omega_1^{(p-r-1)}(x) \\
&\qquad\qquad + (p-r-1)(t_{p+1}-x)M_1^{(r)}(x)\,\omega_1^{(p-r-2)}(x)\Big].
\end{aligned}$$

Or, in terms of A, B,

$$B_{p,r} \le \frac{1}{p-r}\Big[(\|D_1\|+\|D_2\|)A_{p-1,r} + \|D_3\|(p-r-1)B_{p-1,r}\Big],$$

where

$$\begin{aligned}
D_1(x) &= \frac{|x-t_0||x-t_p|}{(t_{p+1}-t_0)(t_p-t_0)}\,\chi_{[t_0,t_p]}(x) \le \frac{1}{4}, \\
D_2(x) &= \frac{|t_{p+1}-x||x-t_1|}{(t_{p+1}-t_0)(t_{p+1}-t_1)}\,\chi_{[t_1,t_{p+1}]}(x) \le \frac{1}{4}, \\
D_3(x) &= \frac{|x-t_0|\,\chi_{[t_0,t_p]}(x) + |t_{p+1}-x|\,\chi_{[t_1,t_{p+1}]}(x)}{t_{p+1}-t_0} \le 1.
\end{aligned}$$

That is,

$$B_{p,r} \le \frac{1}{p-r}\left[\frac{1}{2}A_{p-1,r} + (p-r-1)B_{p-1,r}\right]. \tag{5.6}$$

3) In (5.6) let us make the changes $(p,r) \to (p-1, r-1)$, and write it down together with (5.5).

$$\begin{aligned} A_{p,r} &\le A_{p-1,r-1} + 2(p-r)\, B_{p-1,r-1} \\ B_{p-1,r-1} &\le \frac{1}{p-r}\left[\frac{1}{2}A_{p-2,r-1} + (p-r-1)\, B_{p-2,r-1}\right]. \end{aligned}$$

The values $A_{p,0}$ could be evaluated directly:

$$A_{p,0} := \sup_t \frac{t_{p+1}-t_0}{p+1}\, \|M\,\omega^{(p)}\| = \sup_t \frac{t_{p+1}-t_0}{p+1}\, \|M\| \le 1.$$

If we put

$$\begin{aligned} a_{p,0} &= 1, \\ a_{p,r} &= a_{p-1,r-1} + 2(p-r)\, b_{p-1,r-1}, \\ b_{p-1,r-1} &= \frac{1}{p-r}\left[\frac{1}{2}a_{p-2,r-1} + (p-r-1)\, b_{p-2,r-1}\right], \end{aligned} \tag{5.7}$$

then

$$A_{p,r} \le a_{p,r}, \quad B_{p,r} \le b_{p,r}.$$

Comparing the expressions for a and b in (5.7), we see that $b_{p-1,r-1} = \frac{1}{2(p-r)}\, a_{p-1,r}$. Hence, (5.7) is reduced to the relations

$$\begin{aligned} a_{p,0} &= 1, \\ a_{p,r} &= a_{p-1,r-1} + a_{p-1,r}, \end{aligned}$$

which define the binomial coefficients $a_{p,r} = \binom{p}{r}$.

Thus,

$$A_{p,r} \le \binom{p}{r},$$

and, respectively,

$$\sup_\omega \frac{t_k - t_0}{k}\, \|M_\omega^{(m-1)}\omega^{(k-m)}\| =: A_{k-1,m-1} \le \binom{k-1}{m-1}$$

which was to be proved.

References

[1] C. de Boor, On local linear functionals which vanish at all B-splines but one, *in* "Theory of Approximation with Applications" (A.G.Law and B.N.Sahney, Eds.), pp. 120-145, Academic Press, New York, 1976.

[2] C. de Boor, The exact condition of the B-spline basis may be hard to determine, *J. Approx. Theory*, 60 (1990), 344-359.

[3] C. de Boor, Splines as linear combinations of B-splines. A survey, *in* "Approximation Theory II" (G.G.Lorentz et al, Eds.), pp. 1-49, Academic Press, New York, 1976.

[4] T. Lyche, A note on the condition number of the B-spline basis, *J. Approx. Theory*, 22 (1978), 202-205.

[5] K. Scherer, The condition number of B-splines and related constants, *in* "Open Problems in Approximation Theory" (B.Bojanov, Ed.), pp.180-191, SCT Publishing, Singapore, 1994.