New upper bound for the B-spline basis condition number II. A proof of de Boor's 2^k-conjecture

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For the p-norm condition number $\kappa_{k,p}$ of the B-spline basis of order k we prove the upper estimate

$$\kappa_{k,p} < k\,2^k.$$

This proves de Boor's 2^k-conjecture up to a polynomial factor.

1. Introduction

It is of central importance for working with B-spline series that its condition number is bounded independently of the underlying knot sequence. This fact was proved by C. de Boor in 1968 for the sup-norm and in 1973 for any L_p-norm (see [B1] for references). In the paper [B2] he gave the direct estimate

$$\kappa_{k,p} < k\,9^k \tag{1.1}$$

for $\kappa_{k,p}$, the worst possible condition number with respect to the p-norm of a B-spline basis of order k, and conjectured that the real value of $\kappa_{k,p}$ grows like 2^k:

$$\kappa_{k,p} \sim 2^k, \tag{1.2}$$

which is seen to be far better than (1.1).

The conjecture was based on numerical calculations of some related constants which moreover gave some evidence that the extreme case occurs for a knot sequence without

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interior knots (the so-called Bernstein knots). Maybe due to this reason, a few papers devoted to the 2^k-conjecture for $\kappa_{k,p}$ were concerned only with the "Bernstein knots" conjecture for the extreme knot sequence, see [B3], [C], [Ly], [S].

These papers gave further support for de Boor's conjecture (1.2), in particular T. Lyche [Ly] obtained a lower bound for $\kappa_{k,\infty}$ from which it follows [S] that

$$\kappa_{k,p} > ck^{-1/p}2^k. \tag{1.3}$$

In the unpublished manuscript [SS1] we returned to de Boor's direct approach in [B2], and considered the possibility of improving his 9^k-estimate by several modifications of his method. In particular, a slight revision based on Kolmogorov's estimate for intermediate derivatives had shown that

$$\kappa_{k,p} < k\gamma^k, \quad \gamma = 6.25.$$

In the previous paper [SS2] we developed a further approach to obtain

$$\kappa_{k,p} < k^{1/2}4^k.$$

In this paper using the same approach we give a surprisingly short and elementary proof of

THEOREM 1. *For all k and all $p \in [1,\infty]$,*

$$\kappa_{k,p} < k\,2^k. \tag{1.4}$$

With respect to (1.2)-(1.3), this confirms C. de Boor's conjecture up to a polynomial factor.

We show also that the optimal factor which can be obtained in (1.4) within this approach is $k^{1/2}$ and discuss further possible approaches by which this factor could be removed.

2. Condition number and related constants

Let $\{\hat N_j\}$ be the B-spline basis of order k on a knot sequence $t = (t_j)$, $t_j < t_{j+k}$, normalized with respect to the L_p-norm ($1 \le p \le \infty$), i.e.,

$$\hat N_j(x) = (k/(t_{j+k} - t_j))^{1/p} N_j(x),$$

where $\{N_j\}$ is the B-spline basis which forms a partition of unity. Recall here that

$$N_j(t) = ([t_j, \ldots, t_{j+k-1}] - [t_{j+1}, \ldots, t_{j+k}])\,(\cdot - t)_+^{k-1}$$

and that

$$N_j(x) > 0, \quad x \in (t_j, t_{j+k}); \quad N_j(x) = 0, \quad x \notin [t_j, t_{j+k}]; \qquad \sum N_j = 1.$$

The condition number of the L_p-normalized basis $\{\hat{N}_j\}$ is defined as

$$\begin{aligned}\kappa_{k,p,t} &:= \sup_b \frac{\|b\|_{l_p}}{\|\sum b_j \hat{N}_j\|_{L_p}} \sup_b \frac{\|\sum b_j \hat{N}_j\|_{L_p}}{\|b\|_{l_p}} \\ &= \sup_b \frac{\|b\|_{l_p}}{\|\sum b_j \hat{N}_j\|_{L_p}},\end{aligned}$$

where the L_p-norm is taken with respect to the smallest interval containing the knot sequence (t_i).

The last equality in the above definition follows from normalization

$$\hat{N}_j(x) = M_j^{1/p}(x) N_j^{1/q}(x), \quad M_j(x) := \frac{k}{t_{j+k} - t_j} N_j(x), \quad \int M_j(x)\, dx = 1,$$

so that

$$\begin{aligned}\|\sum b_j \hat{N}_j\|_{L_p} &= \|\sum b_j M_j^{1/p} N_j^{1/q}\|_{L_p} \;\leq\; \|(\sum b_j^p M_j)^{1/p} (\sum N_j)^{1/q}\|_{L_p} \\ &= \|(\sum b_j^p M_j)^{1/p}\|_{L_p} \;=\; \|\sum b_j^p M_j\|_{L_1}^{1/p} \\ &\leq \|b\|_{l_p},\end{aligned}$$

with equalities for $b_j = ((t_{j+k} - t_j)/k)^{1/p}$.

The worst B-spline condition number is defined then as

$$\kappa_{k,p} := \sup_t \kappa_{k,p,t}.$$

Its value gives a measure for the uniform stability of the B-spline basis and is important for numerical calculations with B-splines.

Following [B2] we introduce now related constants that are upper bounds for $\kappa_{k,p}$. This has been done already in [SS2] but for convenience of the reader we state here again the relevant lemmas. More details can be found in [B1], [B2] and [S].

LEMMA A. *Let H_i be the class of functions $h \in L_q$ such that*

$$\begin{aligned}&1) \quad \operatorname{supp} h \subset [t_i, t_{i+k}] \\ &2) \quad \textstyle\int h N_j = \delta_{ij}\end{aligned}$$

and let

$$D_{k,p} := \sup_t \sup_i \inf_{h \in H_i} \left\{ (t_{i+k} - t_i)^{1/p} \|h\|_q \right\}$$

where $1/p + 1/q = 1$. Then

$$\kappa_{k,p} \leq D_{k,p}.$$

Now set

$$\psi_i(x) := \frac{1}{(k-1)!} \prod_{\nu=1}^{k-1} (x - t_{i+\nu}).$$

Then an easy way for obtaining $h \in H_i$ is to set $h = (g\psi_i)^{(k)}$ for some appropriate smooth function g. We formulate this as

Lemma B. *Let G_i be the class of functions g such that*

$$1) \quad g\psi_i \in W_q^k[t_i, t_{i+k}],$$

$$2) \quad g\psi_i = \begin{cases} 0, & k\text{-fold at } t_i, \\ \psi_i, & k\text{-fold at } t_{i+k}, \end{cases}$$

and let $G_i^{(k)} := \{(g\psi_i)^{(k)} : \ g \in G_i\}$. Then

$$G_i^{(k)} \subset H_i.$$

Combining Lemmas A and B gives

Corollary.

$$\kappa_{k,p} \le B_{k,p} := \sup_t \sup_i \inf_{g \in G_i} \left\{ (t_{i+k} - t_i)^{1/p} \|(g\psi_i)^{(k)}\|_q \right\}.$$

Finally, due to the local character of the quantity $B_{k,p}$, it is sufficient to restrict attention to the meshes Δ of the form

$$\Delta = (t_0 \le t_1 \le \ldots \le t_k), \quad t_0 < t_k.$$

Set also

$$\omega(x) := \omega_\Delta(x) = \frac{1}{(k-1)!} \prod_{i=1}^{k-1} (x - t_i) = \psi_0(x), \tag{2.1}$$

and

$$N(t) = N_\Delta(t) = ([t_0, \ldots, t_{k-1}] - [t_1, \ldots, t_k]) (\cdot - t)_+^{k-1}.$$

Lemma C. *For ω given via Δ as in* (2.1)*, let G_Δ be the class of functions g such that*

$$1) \quad g\ \omega \in W_q^k[t_0, t_k]$$

$$2) \quad g\ \omega = \begin{cases} 0, & k\text{-fold at } t_0, \\ \omega, & k\text{-fold at } t_k, \end{cases}$$

and let

$$B_{k,p} := \sup_\Delta \inf_{g \in G_\Delta} (t_k - t_0)^{1/p} \|(g\ \omega)^{(k)}\|_q.$$

Then

$$\kappa_{k,p} \le B_{k,p} \le B_{k,1}. \tag{2.2}$$

Remark. Lemma A is taken from [B2, p.123] whereas Lemmas B and, respectively, C are somewhat more accurate versions of what is given in [B2, Eq.(4.1)]. Namely, they show the possibility to choose a smoothing function g *depending* on ω. C. de Boor's estimate of $B_{k,1}$ resulting in (1.1) was based on the inequalities

$$\begin{aligned} B_{k,1} &\le \inf_{g\in G_\Delta} \sup_\omega \|(g\omega)^{(k)}\|_\infty \le \inf_{g\in G_\Delta} \sum_{i=m}^k \binom{k}{m} \|g^{(m)}\|_\infty \sup_\omega \|\omega^{(k-m)}\|_\infty \\ &\le \sum_{i=m}^k \binom{k}{m} \|g_*^{(m)}\|_\infty \sup_\omega \|\omega^{(k-m)}\|_\infty, \end{aligned}$$

with some special choice of $g_* \in G := \cap G_\Delta$ that is seen to be *independent* of ω. Notice, that in the latter sum for any choice of $g_* \in G$ the term with $m = k$ is equal at least to 4^{k-1} (see [B2,p.132]).

3. Proof of Theorem 1.

The idea in the previous paper [SS2] was to choose $g \in G_\Delta$ as the indefinite integral of the L_1-normalized B-spline, i.e.,

$$g_\Delta(x) := \frac{k}{t_k - t_0} \int_{t_0}^x N_\Delta(t)\, dt.$$

Then, the inclusion $g_\Delta \in G_\Delta$ is almost evident (see [SS2]), and thus we can majorize the constant $B_{k,1}$ by

$$B_{k,1} \le S_{k,1} := \sup_\Delta\, (t_k - t_0) \|s_\Delta^{(k)}\|_\infty \tag{3.1}$$

where

$$s_\Delta := g_\Delta\, \omega_\Delta. \tag{3.2}$$

Notice that supp $s_\Delta^{(k)} \subset [t_0, t_k]$, so that actually the L_∞-norm in (3.1) is taken over $[t_0, t_k]$.

In view of

$$(t_k - t_0)\, s_\Delta^{(k)}(x) = k \sum_{m=1}^k \binom{k}{m} N_\Delta^{(m-1)}(x)\, \omega_\Delta^{(k-m)}(x), \tag{3.3}$$

we showed in [SS2] that, for any Δ and $m = 1, \ldots, k$,

$$\|N_\Delta^{(m-1)}\, \omega_\Delta^{(k-m)}\|_\infty \le \binom{k-1}{m-1}, \tag{3.4}$$

which, by Lemma C and (3.1)-(3.3), implies the bound

$$\kappa_{k,p} < k^{1/2} 4^k.$$

Here we improve (3.4) by

LEMMA 1. *For any Δ, and $m = 1, \ldots, k$*

$$\|N_\Delta^{(m-1)}\, \omega_\Delta^{(k-m)}\|_\infty \le 1. \tag{3.5}$$

Now, by (3.1)-(3.5) and Lemma C,

$$\kappa_{k,p} \le S_{k,1} \le k \sum_{m=1}^{k} \binom{k}{m} = k(2^k - 1) < k2^k$$

which proves Theorem 1.

Remark. If ω has a multiple zero

$$\tau_\nu := t_{\mu_\nu} = t_{\mu_\nu+1} = \ldots = t_{\mu_\nu+p_\nu-1}$$

of multiplicity p_ν, then $N_\Delta^{(k-p_\nu)}$ has a jump at τ_ν. In this case we can define the value $N_\Delta^{(k-p_\nu+q)}(\tau_\nu)\omega^{(p_\nu-1-q)}(\tau_\nu)$ as a limit either from the left or from the right. This limit is equal to zero, if $\tau_\nu \in (t_0, t_k)$. Also this definition justifies the equality (3.3).

4. Lee's formula and a lemma of interpolation

For arbitrary $r \in \mathbf{Z}_+$ and $t \in \mathbf{R}$, set

$$\phi_r(x,t) := \frac{1}{r!}(x-t)_+^r,$$

and define $Q_{\delta_1}(x,t)$ and $Q_{\delta_2}(x,t)$ as algebraic polynomials of degree $k-1$ with respect to x that interpolate the function $\phi_{k-1}(\cdot,t)$ on the meshes

$$\delta_1 = (t_0, t_1, \ldots, t_{k-1}), \quad \delta_2 = (t_1, \ldots, t_{k-1}, t_k),$$

respectively.

The following nice formula is due to Lee [L].

Lemma D [L]. *For any Δ,*

$$N(t)\,\omega(x) = Q_{\delta_1}(x,t) - Q_{\delta_2}(x,t). \tag{4.1}$$

Proof [L]. The difference on the right-hand side is an algebraic polynomial of degree $k-1$ with respect to x that is equal to zero at $x = t_1, \ldots, t_{k-1}$, hence

$$Q_{\delta_1}(x,t) - Q_{\delta_2}(x,t) = c(t) \prod_{i=1}^{k-1}(x - t_i).$$

Further, since the leading coefficient of the Lagrange interpolant to f on the mesh $(\tau_i)_{i=1}^k$ is equal to $[\tau_1, \ldots, \tau_k]f$, we have

$$c(t) = ([t_0, \ldots, t_{k-1}] - [t_1, \ldots, t_k])\,\phi_{k-1}(\cdot,t) =: \frac{1}{(k-1)!}N(t),$$

and the lemma is proved.

We will use Lee's formula (4.1) to evaluate the product $N^{(m-1)}(t)\omega^{(k-m)}(t)$ by taking the corresponding partial derivatives with respect to x and t in (4.1) and setting $x = t$.

Our next two lemmas give a bound for the values obtained in that way on the right-hand side of (4.1).

For arbitrary $p \in \mathbf{N}$, $p \geq r$, and any sequence

$$\delta = (\tau_0 \leq \tau_1 \leq \ldots \leq \tau_p),$$

define, for a fixed t,

$$Q_t(x) := Q(x,t) := Q(x,t;\phi_r,\delta)$$

as the polynomial of degree p with respect to x that interpolates $\phi_r(\cdot,t)$ at δ.

Lemma 2. *For any admissible p, r, t, δ,*

$$0 \leq Q_t^{(r)}(x)\Big|_{x=t} \leq 1 \tag{4.2}$$

where the derivative is taken with respect to x.

Proof. First we prove

A. The case $r = 0$. Then $Q_t(\cdot)$ is a polynomial of degree $\leq p$ that interpolates, for this fixed t, the function

$$(x-t)_+^0 := \begin{cases} 1, & x \geq t; \\ 0, & x < t. \end{cases}$$

We have to prove that

$$0 \leq Q_t(x)\Big|_{x=t} \leq 1 \tag{4.3}$$

and distinguish the following cases:

A1. If $t = \tau_i$ for some i, then (4.3) is evident.

A2. If all the points of interpolation lie either to the left or to the right of t, i.e., if

$$\tau_p < t, \quad \text{or} \quad t < \tau_0,$$

then

$$Q_t \equiv 0, \quad \text{or} \quad Q_t \equiv 1,$$

respectively, and (4.3) holds.

A3. If t lies between two points, i.e., for some ν

$$\tau_0 \leq \ldots \leq \tau_\nu < t < \tau_{\nu+1} \leq \ldots \leq \tau_p,$$

then in view of $Q_t'(x) = [Q_t - \phi_0(\cdot,t)]'(x)$ for $x \neq t$, the polynomial $Q_t'(x)$ has at least ν zeros on the left of τ_ν, and at least $p - \nu - 1$ zeros on the right of $\tau_{\nu+1}$, which gives $p - 1$ zeros in total. Hence Q_t' has no zeros in $(\tau_\nu, \tau_{\nu+1})$, so that Q_t is monotone in $(\tau_\nu, \tau_{\nu+1})$, that is

$$0 = Q_t(\tau_\nu) < Q_t(t) < Q_t(\tau_{\nu+1}) = 1.$$

B. The case $r > 0$. This case is reduced to the case $r = 0$ by Rolle's theorem. The difference $\phi_r(\cdot, t) - Q_t$ has $p + 1$ zeros (counting multiplicity), thus its r-th derivative $\phi_0(\cdot, t) - Q_t^{(r)}$ must have at least $p + 1 - r$ changes of sign.

If (4.2) does not hold, then this function does not change sign at $x = t$, and $Q_t^{(r)}$ is a polynomial of degree $p - r$ that interpolates $\phi_0(\cdot, t)$ at $p - r + 1$ points all distinct from t. But according to the Case A3 this would imply (4.2), a contradiction.

Hence, (4.2) holds, and the lemma is proved.

LEMMA 3. *For any admissible p, r, t, δ,*

$$0 \le (-1)^s \frac{\partial^{r-s}}{\partial x^{r-s}} \frac{\partial^s}{\partial t^s} Q(x, t)\Big|_{x=t} \le 1. \tag{4.4}$$

Proof. Let l_i be the fundamental Lagrange polynomials of degree p for the mesh δ, i.e., $l_i(\tau_j) = \delta_{ij}$. Then $Q_t = Q(\cdot, t)$, which is the Lagrange interpolant to $\phi_r(\cdot, t)$, can be expressed as

$$Q(x, t) = \frac{1}{r!} \sum_{i=0}^{p} (\tau_i - t)_+^r \, l_i(x).$$

Thus, we obtain

$$(-1)^s \frac{\partial^s}{\partial t^s} Q(x, t) = \frac{1}{(r-s)!} \sum_{i=0}^{p} (\tau_i - t)_+^{r-s} \, l_i(x).$$

It is readily seen that

$$Q_{0,t}(x) := Q_0(x, t) := (-1)^s \frac{\partial^s}{\partial t^s} Q(x, t)$$

is a polynomial of degree p with respect to x that interpolates

$$\phi_{r-s}(\cdot\,, t) = \frac{1}{(r-s)!} \, (\cdot - t)_+^{r-s}$$

at the same mesh δ. Now (4.4) follows from Lemma 2.

5. Proof of Lemma 1

We need to bound

$$N^{(s)}(t)\, \omega^{(k-1-s)}(t) = N^{(s)}(t)\, \omega^{(k-1-s)}(x)\Big|_{x=t}, \qquad s = 0, 1, \ldots, k-1.$$

Now according to Lemma D

$$N^{(s)}(t)\, \omega^{(k-1-s)}(x) = \frac{\partial^{k-1-s}}{\partial x^{k-1-s}} \frac{\partial^s}{\partial t^s} Q_{\delta_1}(x, t) - \frac{\partial^{k-1-s}}{\partial x^{k-1-s}} \frac{\partial^s}{\partial t^s} Q_{\delta_2}(x, t),$$

and by Lemma 3 for any δ

$$0 \le (-1)^s \frac{\partial^{k-1-s}}{\partial x^{k-1-s}} \frac{\partial^s}{\partial t^s} Q_\delta(x, t)\Big|_{x=t} \le 1.$$

Hence, since both terms are of the same sign and of absolute value ≤ 1,

$$\left| N^{(s)}(t) \cdot \omega^{(k-1-s)}(t) \right| \le 1,$$

which proves Lemma 1.

6. On the factor k in Theorem 1

Numerical computations [B3] show that

$$\kappa_{k,p} \le c\, 2^k, \tag{7.1}$$

so a natural question is whether the factor k in the bound

$$\kappa_{k,p} < k\, 2^k \tag{7.2}$$

of Theorem 1 can be removed.

A simple example will show now that within the particular method we used in Section 3 (see (3.1)), an extra polynomial factor $\sqrt{k}$ appears unavoidably. Namely, one can prove that for some choice of Δ_*

$$S_{k,1} \ge (t_k - t_0)\, \|s^{(k)}_{\Delta_*}\|_\infty \ge c k^{1/2}\, 2^k.$$

In fact, in the case of the Bernstein knots Δ_ν in $[0,1]$, i.e., for

$$\omega_\nu(x) = \frac{1}{(k-1)!} x^\nu (x-1)^{k-1-\nu},$$

we have

$$N_\nu(x) = \binom{k-1}{\nu} x^{k-1-\nu}(1-x)^\nu,$$

and obtain

$$\begin{aligned} s^{(k)}_{\Delta_\nu}(x) &= \frac{k}{(k-1)!}\binom{k-1}{\nu} \\ &\quad \times \sum_{m=1}^{k} \binom{k}{m} \left[x^{k-1-\nu}(1-x)^\nu\right]^{(m-1)} \left[x^\nu (x-1)^{k-1-\nu}\right]^{(k-m)}. \end{aligned}$$

It is not hard to see that at $x = 1$ the m-th term vanishes, unless $m = \nu + 1$, which gives

$$|s^{(k)}_{\Delta_\nu}(1)| = \frac{k}{(k-1)!}\binom{k-1}{\nu} \cdot \binom{k}{\nu+1} \nu!(k-1-\nu)! = k\binom{k}{\nu+1}.$$

With this, we take $\nu_* + 1 = \lfloor k/2 \rfloor$ to obtain

$$|s_{\Delta_*}(1)| = k\binom{k}{\lfloor k/2 \rfloor} > c k^{1/2} 2^k.$$

7. Possible refinements

We describe here some further approaches that may permit removal of the polynomial factor in the upper bound for the sup-norm condition number $\kappa_{k,\infty}$.

1. The first approach is to majorize $\kappa_{k,\infty}$ using the intermediate estimate (2.2) with the value $B_{k,\infty}$ instead of $B_{k,1}$ used in Theorem 1, that is

$$\kappa_{k,\infty} \le B_{k,\infty}.$$

Then the desired 2^k-bound without an extra factor will follow from the following

CONJECTURE. *For any $\omega = \omega_\Delta$, there exists a function $g_* \in G_\Delta$ such that*

$$\operatorname{sign}\, g_*^{(m)}(x) = \operatorname{sign}\, \omega^{(k-m)}(x), \quad x \in [t_0, t_k], \quad m = 1, \ldots, k. \tag{7.1}$$

This conjecture implies that

$$\|g_*^{(m)}\, \omega^{(k-m)}\|_{L_1[t_0,t_k]} = \left| \int_{t_0}^{t_k} g_*^{(m)}(x)\, \omega^{(k-m)}(x)\, dx \right|.$$

Then observe that, because of the boundary conditions satisfied by g_* and the way g_* and ω_Δ are normalized,

$$(-1)^m \int_{t_0}^{t_k} g_*^{(m)}(x)\, \omega^{(k-m)}(x)\, dx = \int_{t_0}^{t_k} g_*'(x)\, \omega^{(k-1)}(x)\, dx = 1.$$

Hence

$$\|g_*^{(m)}\, \omega^{(k-m)}\|_{L_1[t_0,t_k]} = 1, \quad m = 1, \ldots, k, \tag{7.2}$$

and using this bound, one could show, exactly as in Section 3, that

$$\kappa_{k,\infty} \le B_{k,\infty} \le \sum_{m=1}^{k} \binom{k}{m} = 2^k - 1.$$

Remark. A function g_* satisfying (7.1) should in a sense be close to the function g_Δ considered in Section 3 (though it is not necessarily unique). Moreover, g_Δ can serve as g_* for the polynomials ω_{Δ_ν} with the Bernstein knots

$$\omega_{\Delta_\nu}(x) = c(x - t_0)^\nu (x - t_k)^{k-1-\nu}.$$

Also, it looks quite probable that, even though the equality (7.2) is not valid with $g_* = g_\Delta$ for arbitrary Δ, there holds

$$\|g_\Delta^{(m)} \omega_\Delta^{(k-m)}\|_{L_1[t_0,t_k]} \le c, \quad m = 1, \ldots, k,$$

that is for the B-spline $M_\Delta(x) = (k/(t_k - t_0))\, N_\Delta(x)$ we have

$$\|M_\Delta^{(m-1)}\omega_\Delta^{(k-m)}\|_{L_1[t_0,t_k]} \le c.$$

2. Another possibility to improve the result of Theorem 1 would be to find a sharp bound for one of the related constants considered in [S] . In this respect it is known, e.g., that

$$\kappa_{k,\infty} \le E_{k,p}^{-1} \tag{7.3}$$

where

$$E_{k,p} := \inf_\Delta \inf_j \inf_{c_i} \{\|N_j - \sum_{i\ne j} c_i N_i\|_p\}.$$

In particular, there is equality in (7.3) for $p=\infty$.

The hope is to prove that the knot sequence at which the value $E_{k,p}$ is attained for $p=1$ or $p=2$ is the Bernstein one, in which case the inequalities

$$E_{k,1}^{-1} < c2^k, \quad \text{or} \quad E_{k,2}^{-1} < c2^k$$

would follow. (It is known that the Bernstein knot sequence is not extreme for $p=\infty$, see [B3]).

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