On L_p-boundedness of the L_2-projector onto finite element spaces

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1. Introduction

In [Sh] we proved that, for p sufficiently close to 2, the L_p-norm of the L_2-projector P onto the space S of polynomial splines of order k on a knot-sequence Δ is bounded independently of Δ:

$$\sup_{\Delta} \|P_{S_k(\Delta)}\|_p \le c_k, \qquad \left|\frac{1}{p} - \frac{1}{2}\right| < \epsilon_k, \qquad \epsilon_k = o(1). \tag{1.1}$$

This was thought to be a support to de Boor's conjecture [B] which states such a boundedness for $p = \infty$ (thus, for all $p \in [1, \infty]$), and which has been proved so far only for some particular k.

The proof itself was rather technical and employed specific properties of the polynomial splines.

In this paper we prove two theorems which show that, on its own, the estimate (1.1) has nothing to do either with the splines, or with de Boor's conjecture.

Namely, in Theorem 1, by elementary means, we show that, for p from a small neighbourhood of $p = 2$, the mesh independent L_p-estimate (1.1) is not something extraordinary, but holds for the L_2-projector onto arbitrary space with a well conditioned basis of finitely supported functions. The *minimal* radius of this neighbourhood depends only on the basis local condition number.

Using Theorem 1 and a known bound for the B-spline condition we give an explicit expression for ϵ_k in (1.1), which we did not manage to compute in [Sh].

On the other hand, in Theorem 2, we construct a sequence of f.e. bases with uniformly bounded local condition numbers, such that the corresponding L_2-projectors are not uniformly L_p-bounded, if p does not belong to a certain neighbourhood of $p = 2$. As local condition (necessarily) tends to ∞, the *maximal* radius of this neighbourhood becomes arbitrary small.

Since the local condition of the B-spline basis of order k grows like 2^k, the latter theorem shows that in proving de Boor's conjecture for large k one should use something more delicate than the mesh independent boundedness of the B-spline condition number.

2. Results

Let $k, n \in \mathbb{N}$, and let $\Delta = \{t_i\}_{i=1}^{n+k}$ be a knot sequence, such that

$$a = t_1 \le t_2 \le \ldots \le t_{n+k} = b, \quad t_i < t_{i+k}.$$

For k, n, Δ as above, and for $d \in \mathbb{R}$, define $\Phi_{k,d}(\Delta)$ as the set of all f.e. bases $\Phi = \{\phi_i\}_{i=1}^n$ with the following properties:

(A_1) $\quad \operatorname{supp} \phi_i = E_i := [t_i, t_{i+k}]$;

(A_2) $\quad \|\phi_i\|_\infty \le 1/k$;

(A_3) $\quad d^{-1}|a_j| \le |E_j|^{-1/p} \|\sum_{i=1}^n a_i \phi_i\|_{L_p(E_j)} \qquad \forall a = (a_i)_{i=1}^n, \quad \forall p \in [1, \infty]$.

For a linear space S we write $S \in S_{k,d}(\Delta)$, if $S = \mathrm{span}\,\{\Phi\}$ with some $\Phi \in \Phi_{k,d}(\Delta)$.

Remarks. 1) Property (A_3) is equivalent to

$$d_p(\Phi) \le d,$$

where

$$d_p(\Phi) := \Big\{ \inf_j \big[|E_j|^{-1/p} \mathrm{dist}\,\big(\phi_j, \mathrm{span}\,\{\phi_i\}_{i \ne j}\big)_{L_p(E_j)}\big]\Big\}^{-1}$$

can be viewed as a *local* p-condition number of Φ. By Hölder inequality,

$$d_p(\Phi) \le d_1(\Phi), \quad \text{if} \quad 1 < p \le \infty,$$

so that it is enough to require (A_3) only for $p = 1$.

2) Also, properties (A_1)–(A_3) provide the estimate

$$\kappa_p(\Phi) \le d_p(\Phi) \le d$$

for the global condition number κ_p of the properly L_p-normalized basis Φ (see §3).

Now let P_S be the operator of the L_2-projection onto S with respect to the usual inner product $(f, g) = \int fg$, i.e.,

$$(f, \sigma) = (P_S f, \sigma), \quad \forall \sigma \in S.$$

For $S \in S_{k,d}(\Delta)$, due to (A_2), this operator maps L_p onto L_p, and we are interested in bounds for its norm

$$\|P_S\|_p = \sup_{\|f\|_p = 1} \|P_S(f)\|_p.$$

The main results of this paper are the following.

Theorem 1. *For any $k \in \mathbb{N}$, $d \in \mathbb{R}$, $d \ge k$,*

$$\sup_{\Delta} \sup_{S \in S_{k,d}(\Delta)} \|P_S\|_p \le c(k, d), \qquad \left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{2kd^2 \ln d}.$$

Theorem 2. *For $k = 2$, and any $d \ge 16$,*

$$\sup_{\Delta} \sup_{S \in S_{2,d}(\Delta)} \|P_S\|_p = \infty, \qquad \left|\frac{1}{p} - \frac{1}{2}\right| > \frac{3}{\sqrt{d}}.$$

The proof of Theorem 1 is given in §§ 3-7 among which only §5 and §7 contain some new results. §§ 3,4,6 are just a rewriting of the known results for the B-spline basis to the f.e. bases satisfying (A_1)-(A_3) and are given for the sake of completeness.

Theorem 1 has just the same "spline-eliminating" character as Demko's result on the inverses of the band matrices, and Descloux's result on the f. e. matrices.

For the spline space $S_k(\Delta)$ the local condition of the B-spline basis $\Phi_k(\Delta)$ of order k satisfies

$$k^{-1} 2^k < \sup_{\Delta} d_1(\Phi_k(\Delta)) \le k 2^k.$$

Thus, by Theorem 1, we obtain

Corollary 1. *For the L_2-projector onto the spline space $S_k(\Delta)$*

$$\sup_{\Delta} \|P_{S_k(\Delta)}\|_p \le c(k, d), \qquad \left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{4^{k+1} k^4}.$$

On the other hand, Theorem 2 shows that the growth of the B-spline basis condition number is not its best property.

3. Condition number

For $1 \le p \le \infty$, set

$$\phi_{i,p} := (k/|E_i|)^{1/p}\phi_i, \quad \Phi_p = \{\phi_{i,p}\}_{i=1}^n. \tag{3.1}$$

and define the p-condition number of Φ as

$$\kappa_p(\Phi) := \sup_{a_j} \frac{\|\sum_j a_j\phi_{j,p}\|_{L_p}}{\|(a_j)\|_{l_p}} \sup_{b_j} \frac{\|(b_j)\|_{l_p}}{\|\sum_j b_j\phi_{j,p}\|_{L_p}}.$$

Lemma 3.1. *If $\Phi \in \Phi_{k,d}(\Delta)$, then*

$$d^{-1}\|a\|_{l_p} \le \|\sum_{i=1}^n a_i\phi_{i,p}\|_{L_p} \le \|a\|_{l_p}, \tag{3.2}$$

i.e.,

$$\kappa_p(\Phi) \le d.$$

Proof. 1) The upper estimate.

1a) If $p = \infty$, then at any $x \in [a,b]$, due to (A_1), at most k functions of Φ differ from zero, what by (A_2) implies

$$\sum_{i=1}^m |\phi_i(x)| \le k \max_i |\phi_i(x)| \le 1, \quad \forall x \in [a,b], \tag{3.3}$$

and respectively,

$$\|\sum_{i=1}^n a_i\phi_i\|_{L_\infty} \le \max_i |a_i| \cdot \|\sum_{i=1}^m \phi_i\|_{L_\infty} \le \max_i |a_i|.$$

1b) For $p = 1$, by (A_2),

$$\|\phi_{i,1}\|_1 := k\,|E_i|^{-1}\|\phi_i\|_1 \le k\,\|\phi_i\|_\infty \le 1, \qquad \forall i,$$

thus,

$$\|\sum_{i=1}^n a_i\phi_{i,1}\|_1 \le \sum_{i=1}^n |a_i|\,\|\phi_{i,1}\|_1 \le \sum_{i=1}^n |a_i|. \tag{3.4}$$

1c) For $1 < p < \infty$, since

$$\phi_{i,p} = \phi_{i,1}^{1/p} \cdot \phi_{i,\infty}^{1/q},$$

by Hölder inequality, and due to (3.3)-(3.4),

$$\begin{aligned} \|\textstyle\sum_{i=1}^n a_i\phi_{i,p}\|_{L_p} &= \|\textstyle\sum a_i\phi_{i,1}^{1/p} \cdot \phi_{i,\infty}^{1/q}\|_{L_p} \le \|(\sum |a_i|^p|\phi_{i,1}|)^{1/p}(\sum |\phi_{i,\infty}|)^{1/q}\|_{L_p} \\ &\le \|(\textstyle\sum a_i^p\phi_{i,1})^{1/p}\|_{L_p} = \|\sum a_i^p\phi_{i,1}\|_{L_1}^{1/p} \le \|(a_i^p)\|_{l_1}^{1/p} \\ &= \|a\|_{l_p}. \end{aligned}$$

2) The lower estimate. By (A_3),

$$d^{-p}|a_j|^p \le k^{-1}\|\sum_i a_i k^{1/p}|E_j|^{-1/p}\phi_i\|_{L_p[E_j]}^p = k^{-1}\|\sum_i a_i\phi_{i,p}\|_{L_p[E_j]}^p,$$

thus, for $f = \sum_i a_i \phi_{i,p}$, we have

$$d^{-p} \sum_j |a_j|^p \le k^{-1} \sum_j \|f\|^p_{L_p[E_j]} \le \|f\|^p_{L_p[t_0,t_{n+k}]}.$$

4. L_2-projector and Gram-matrix

Here we will establish equivalence between the L_p-norm of P_S and the l_p-norm of inverse of a Gram-type matrix A_p.

Lemma 4.1. *Let $\Phi \in \Phi_{k,d}(\Delta)$, $S = \mathrm{span}\,\{\Phi\}$, and let A_p be the $n \times n$ matrix*

$$A_p = \{(\phi_{i,q}, \phi_{j,p})\}^n_{i,j=1}.$$

Then

$$d^{-2}\|A_p^{-1}\|_{l_p} \le \|P_S\|_{L_p} \le \|A_p^{-1}\|_{l_p}. \tag{4.1}$$

Proof. We make use of the following nice formula of de Boor [B].

$$\|P_S\|_p = \sup_{s\in S} \inf_{\sigma\in S} \frac{\|s\|_p\|\sigma\|_q}{|(s,\sigma)|} = \sup_{\sigma\in S} \inf_{s\in S} \frac{\|s\|_p\|\sigma\|_q}{|(s,\sigma)|}. \tag{4.2}$$

Now, if

$$s = \sum_{j=1}^n x_j \phi_{j,p}, \quad \sigma = \sum_{i=1}^n y_i \phi_{i,q}.$$

for some sequences $x = (x_j)$, $y = (y_i)$, then

$$(s,\sigma) = \sum_{i=1}^n y_i \sum_{j=1}^n (\phi_{i,q}, \phi_{j,p}) x_j = (A_p x, y), \qquad A_p = \{(\phi_{i,p}, \phi_{j,q})\}^n_{i,j=1},$$

and according to (4.2),

$$P_S = \sup_{s\in S} \inf_{\sigma\in S} \frac{\|s\|_p\|\sigma\|_q}{|(s,\sigma)|} = \sup_{x\in l_p} \inf_{y\in l_q} \frac{\|s\|_{L_p}\|\sigma\|_{L_q}}{|(A_p x, y)|}.$$

By Lemma 3.1,

$$d^{-1}\|x\|_{l_p} \le \|s\|_{L_p} \le \|x\|_{l_p}, \quad d^{-1}\|y\|_{l_q} \le \|\sigma\|_{L_q} \le \|y\|_{l_q},$$

so that

$$d^{-2}\gamma \le \|P_S\|_p \le \gamma, \qquad \gamma = \sup_{x\in l_p} \inf_{y\in l_q} \frac{\|x\|_{l_p}\|y\|_{l_q}}{|(A_p x, y)|} = \|A^{-1}\|_{l_p}$$

The lemma is proved.

Since for any matrix $B = (b_{ij})$ holds

$$|b_{ij}| \le \|B\|_p \le \|B\|_1^{1/p} \|B\|_\infty^{1/q} \le \max\{\|B\|_1, \|B\|_\infty\},$$

a corollary of (4.1) is

Lemma 4.2. *Let $\Phi \in \Phi_{k,d}(\Delta)$, $S = \mathrm{span}\,\{\Phi\}$,*

$$A_p := \{(\phi_{i,q}, \phi_{j,p}\}^n_{i,j=1}, \qquad A_p^{-1} := (b_{ij,p})^n_{i,j=1}$$

Then

$$\|P_S\|_p \geq d^{-2} \max_{ij} |b_{ij,p}|, \tag{4.3}$$

$$\|P_S\|_p \leq \max \left\{ \max_i \sum_{j=1}^n |b_{ij,p}|,\ \max_j \sum_{i=1}^n |b_{ij,p}| \right\}. \tag{4.4}$$

5. Proof of Theorem 1.

In order to apply the estimate (4.4) it sufficies to establish a certain decay of the entry $b_{ij,p}$ as a function of $|i-j|$ and p.

From the normalization condition (3.1), it follows that

$$A_p := \{(\phi_{i,q}, \phi_{j,p}\} = E^{1/p} A_\infty E^{-1/p},$$

with

$$E := \operatorname{diag}(|E_1|, |E_2|, \dots |E_n|).$$

This gives for the entries of the inverse matrices $A_p^{-1} = (b_{ij,p})$ the relation

$$b_{ij,p} = (|E_j|/|E_i|)^{1/p} b_{ij,\infty}. \tag{5.1}$$

and, as a corollary, the intermediate estimate

$$|b_{ij,p}| = |b_{ij,2}|^{2/p} |b_{ij,\infty}|^{1-2/p}. \tag{5.2}$$

As we will show in §§ 6-7,

$$|b_{ij,2}| \leq d^2 \gamma_2^{|i-j|}, \quad \gamma_2 = \left(\frac{d^2-1}{d^2+1}\right)^{1/2(k-1)}, \tag{5.3}$$

$$|b_{ij,\infty}| \leq d^2 \gamma_\infty^{|i-j|}, \quad \gamma_\infty = d/\sqrt{2}, \tag{5.4}$$

so that (5.2) implies

$$|a_{ij,p}| \leq d^2 \gamma_p^{|i-j|}, \quad \gamma_p := \gamma_2^{2/p} \gamma_\infty^{1-2/p}, \quad p \geq 2. \tag{5.5}$$

The inequality $\gamma_p < 1$ means

$$\gamma_2^{2/p} \gamma_\infty^{1-2/p} = \gamma_\infty \left(\frac{\gamma_2}{\gamma_\infty}\right)^{2/p} < 1, \quad p \geq 2,$$

what results in

$$1 \geq 2/p > \frac{\ln \gamma_\infty}{\ln \gamma_\infty + \ln(1/\gamma_2)}$$

i.e.,

$$0 \leq \frac{1}{2} - \frac{1}{p} < \frac{1}{2} \frac{\ln(1/\gamma_2)}{\ln \gamma_\infty + \ln(1/\gamma_2)} =: \epsilon_0. \tag{5.6}$$

Since $\ln x \geq 1 - 1/x$, we have

$$\begin{aligned} \ln(1/\gamma_2) &:= \frac{1}{2(k-1)} \ln\left(\frac{d^2+1}{d^2-1}\right) > \frac{1}{k-1} \frac{1}{d^2+1}, \\ \ln \gamma_\infty &:= \ln(d/\sqrt{2}), \end{aligned}$$

therefore,

$$\epsilon_0 := \frac{1}{2}\frac{1}{1+\frac{\ln\gamma_\infty}{\ln(1/\gamma_2)}} > \frac{1}{2}\frac{1}{1+(k-1)(d^2+1)\ln(d/\sqrt{2})}.$$

From the inequality $d \ge k \ge 2$, it follows that

$$\epsilon_0 > \frac{1}{2}\frac{1}{kd^2\ln d}$$

so that for p satisfying

$$0 \le \frac{1}{2} - \frac{1}{p} \le \epsilon^* := \frac{1}{2}\frac{1}{kd^2\ln d} \tag{5.7}$$

the inequality (5.6) is also satisfied.

This means, that if p belongs to the interval (5.7), then for γ_p defined in (5.5) we have

$$\gamma_p \le \gamma_{p^*} = \gamma_{k,d} < 1,$$

therefore,

$$|b_{ij,p}| \le d^2\gamma_{k,d}^{|i-j|}, \quad \gamma_{k,d} < 1,$$

and, by (4.4),

$$\|P_S\|_p \le c_{k,d}.$$

Finally,

$$\|P_S\|_p \le c_{k,d}, \qquad |\frac{1}{2} - \frac{1}{p}| \le \frac{1}{2kd^2\ln d}.$$

Theorem 1 is proved.

6. Proof of (5.3)

Denote by $\Phi_q^* = \{\phi_{i,q}^*\}$ the basis dual to Φ_q,

$$(\phi_{i,q}, \phi_{j,q}^*) = \delta_{ij}, \qquad i,j = \overline{1,n}.$$

It follows from the definition of $A_p^{-1} := \{(\phi_{i,q}, \phi_{j,p})\}^{-1} := (b_{ij,p})$, that

$$\phi_{i,q}^* = \sum_{j=1}^n b_{ij,p}\phi_{j,p}.$$

Lemma 6.1. *Let $\Phi \in \Phi_{k,d}(\Delta)$, and $\phi_{i,2}^* = \sum_{j=1}^n b_j\phi_{j,2}$. Then*

$$|b_j| \le d^2\gamma_2^{|i-j|}, \quad \gamma_2 = \left(\frac{d^2-1}{d^2+1}\right)^{1/2k}. \tag{5.3}$$

Proof. 1) Since $\phi_{i,2}^*$ is orthogonal to span $\{\phi_{j,2}\}_{j\ne i}$, we have

$$(\phi_{i,2}^*, \phi_{i,2}^*) = (\phi_{i,2},\, b_i\phi_{i,2} + \sum_{j\ne i} b_j\phi_{j,2}) = (\phi_{i,2}^*, b_i\phi_{i,2}) = b_i,$$

i.e.,

$$\|\phi_{i,2}^*\|_2^2 = b_i.$$

On the other hand, by (3.2),

$$d^{-2}\sum_{j=1}^{n} b_j^2 \le \|\phi_{i,2}^*\|_2^2,$$

i.e.,

$$b_i^2 \le \sum_{j=1}^{n} b_j^2 \le d^4. \tag{6.1}$$

2) For $m > i+1-k$, set

$$\psi_m = \sum_{j=m}^{n+k} b_j\phi_j.$$

Then

$$\psi_m(x) = \phi_{i,2}^*(x), \quad x \ge t_{m+k-1}$$

i.e. ψ_m is orthogonal to $\mathrm{span}\{\phi_{i,2}\}_{i\ge m+k-1}$, in particular to ψ_{m+k-1}. This gives

$$\|\psi_m\|_{L_2[a,b]}^2 + \|-\psi_{m+k-1}\|_{L_2[a,b]}^2 = \|\psi_m - \psi_{m+k-1}\|_{L_2[a,b]}^2,$$

i.e.

$$\|\sum_{j\ge m} b_j\phi_{j,2}\|_2^2 + \|\sum_{j\ge m+k-1} b_j\phi_{j,2}\|_2^2 = \|\sum_{j=m}^{m+k-2} b_j\phi_{j,2}\|_2^2,$$

or, with respect to (3.2),

$$d^{-2}\sum_{j\ge m} b_j^2 + d^{-2}\sum_{j\ge m+k-1} b_j^2 \le \sum_{j=m}^{m+k-2} b_j^2 \tag{6.2}$$

Set

$$B_m := \sum_{j\ge m} b_j^2.$$

Then (6.2) reads

$$d^{-2}B_m + d^{-2}B_{m+k-1} \le B_m - B_{m+k-1}$$

or

$$B_{m+k-1} \le \gamma B_m, \quad \gamma := \frac{d^2-1}{d^2+1}.$$

For any $j \ge i+1$, with some $0 \le l_0 \le k-2$, this gives

$$b_j^2 \le B_j \le \gamma^{\lfloor \frac{j-i}{k-1}\rfloor + 1} B_{i-l_0} \le \gamma^{\frac{j-i}{k-1}} d^4$$

Finally,

$$|b_j| \le d^2\gamma_2^{|i-j|}, \quad \gamma_2 = \left(\frac{d^2-1}{d^2+1}\right)^{1/(2k-2)}$$

7. Proof of (5.4)

Lemma 7.1. *Let $\Phi \in \Phi_{k,d}(\Delta)$, and let $\phi_{i,1}^* = \sum_{j=1}^{n} a_j\phi_j$. Then*

$$|a_j| \le d^3\gamma_\infty^{|i-j|}, \quad \gamma_\infty = d/\sqrt{2}. \tag{5.4}$$

Proof. 1) By (5.1),(6.1),

$$|a_i| := |b_{ii,\infty}| = |b_{ii,2}| =: |b_i| \le d^2.$$

2) Let $I_{i+l_0} = [t_{i+l_0}, t_{i+l_0+1}]$ be the largest subinterval of $E_i = [t_i, t_{i+k}]$, i.e.,

$$I_{i+l_0} \subset E_i, \qquad |I_{i+l_0}| \ge k^{-1}|E_i|, \qquad 0 \le l_0 \le k-1.$$

Then, for any $j = 1, \ldots, k$,

$$I_{i+l_0} \subset E_{i+l_0-k+j},$$

and we have

$$\begin{aligned} d^{-1}|a_{i+l_0-k+j}| &\le |E_{i+l_0-k+j}|^{-1/2} \|\phi^*_{i,1}\|_{L_2(E_{i+l_0-k+j})} \\ &\le |I_{i+l_0}|^{-1/2} \|\phi^*_{i,1}\|_{L_2(E_{i+l_0-k+j})} \\ &\le k^{1/2}|E_i|^{-1/2} \|a_i\phi_i\|_{L_2(E_i)} \\ &\le k^{1/2} \|a_i\phi_i\|_{L_\infty(E_i)} \\ &\le k^{-1/2}|a_i| \le k^{-1/2} d^2 \le d^2, \end{aligned}$$

i.e.

$$\max_{i+l_0+1-k \le j \le i+l_0} |a_j| \le d^3. \tag{7.1}$$

3) For $m > i+1-k$ set

$$\psi_m = \sum_{j=m}^{n+k} a_j \phi_j.$$

Then

$$\psi_m(x) = \phi^*_{i,1}(x), \quad x \ge t_{m+k-1}$$

i.e. ψ_m is orthogonal to $\mathrm{span}\{\phi_i\}_{i \ge m+k-1}$, in particular to ψ_{m+k-1}. This gives

$$\|\psi_m\|^2_{L_2[t_{m+k-1},b]} + \|-\psi_{m+k-1}\|^2_{L_2[t_{m+k-1},b]} = \|\psi_m - \psi_{m+k}\|^2_{L_2[t_{m+k-1},b]}.$$

Since

$$E_{m+k-1} = [t_{m+k-1}, t_{m+2k-1}] \subset [t_{m+k-1}, b],$$

$$\mathrm{supp}\,(\psi_m - \psi_{m+k-1}) \cap [t_{m+k-1}, b] = [t_{m+k-1}, t_{m+2k-2}] \subset E_{m+k-1}$$

we also have

$$\|\psi_m\|^2_{L_2(E_{m+k-1})} + \|\psi_{m+k}\|^2_{L_2(E_{m+k-1})} \le \|\psi_m - \psi_{m+k-1}\|^2_{L_2(E_{m+k-1})}.$$

By (A_3) we have

$$\begin{aligned} 2d^{-2}|a_{m+k-1}|^2 &\le |E_{m+k-1}|^{-1}\big(\|\psi_{m+k-1}\|^2_{L_2[E_{m+k-1}]} + \|\psi_m\|^2_{L_2[E_{m+k-1}]}\big) \\ &\le |E_{m+k-1}|^{-1} \|\psi_m - \psi_{m+k-1}\|^2_{L_2[E_{m+k-1}]} \\ &\le \|\psi_m - \psi_{m+k-1}\|^2_{L_\infty[E_{m+k-1}]} \\ &= \|\sum_{j=m}^{m+k-2} a_j\phi_j\|^2_{L_\infty[E_{m+k-1}]} \\ &\le \max_{m \le j \le m+k-2} |a_j|^2, \end{aligned}$$

what implies

$$|a_m| \le (d/\sqrt{2})^{m-i-l_0} \max_{i+l_0+2-k \le j \le i+l_0} |a_j|.$$

Finally, with respect to (7.1) ,

$$|a_m| \le d^3 (d/\sqrt{2})^{|m-i|}.$$

8. Proof of Theorem 2

In this section, for given $\epsilon > 0$ we construct a sequence of bases $\Phi_n = \{\phi_i\}_{i=1}^n$, such that

(A) $\Phi_n \in \Phi_{2,d}(\Delta_n)$, with some Δ_n and some $d = d_\epsilon = O(1/\epsilon^2)$;

(B) for the matrices

$$A_{p,n} := \{(\phi_{i,q}, \phi_{j,p})\}_{i,j=1}^n, \quad 1/p = 1/2 - \epsilon,$$

with some constants c_ϵ and $\xi_\epsilon > 1$ holds

$$\|A_{p,n}^{-1}\|_p > c\,\xi^n.$$

This proves Theorem 2.

1) Let $f_1, f_2 \in L_\infty[0,1]$ be given. For $k = 2$, and

$$\Delta_n = \{a = t_1 < t_2 < \ldots < t_{n+1} < t_{n+2} = b\}, \quad \text{with} \quad h_i := t_{i+1} - t_i,$$

define a f.e. basis $\Phi = \{\phi_i\}_{i=1}^n$ as

$$\phi_i(x) = \begin{cases} f_1\left((x - t_i)/h_i\right), & x \in I_i; \\ f_2\left((x - t_{i+1})/h_{i+1}\right), & x \in I_{i+1}. \end{cases} \tag{8.1}$$

Further, consider the case of the geometric mesh $\Delta_{n,z}$ with the local mesh ratio z, i.e.,

$$h_{i+1}/h_i = z, \quad \forall i = \overline{1, n+1}.$$

As we show in § 9, such a basis $\Phi := \Phi(f_1, f_2, \Delta_{n,z})$ has the following properties.

2) The local L_1-condition of Φ is determined by

$$d_1(\Phi) = \frac{1+z}{e(f_1) + z e(f_2)}, \tag{8.2}$$

where

$$e(f_1) := \inf_\alpha \|f_1 - \alpha f_2\|_{L_1[0,1]}, \qquad e(f_2) := \inf_\beta \|f_2 - \beta f_1\|_{L_1[0,1]}.$$

3) If for $\epsilon \in (0, 1/2]$, and $z > 1$ holds

$$\left(z^\epsilon + z^{-\epsilon}\right) z^{1/2} \int f_1 f_2 > \int f_1^2 + z \int f_2^2, \tag{8.3}$$

then for the inverse of the matrix

$$A_{p,n} := \{(\phi_{i,q}, \phi_{j,p})\}_{i,j=1}^n, \quad 1/p = 1/2 - \epsilon,$$

with some constants c_ϵ and $\xi_\epsilon > 1$

$$\|A_p^{-1}\|_p > |b_{1,n,p}| > c\,\xi^n. \tag{8.4}$$

4) Now, for some $z > 1$ and arbitrary small ϵ, we construct the functions f_1, f_2 satisfying (8.3), such that d_1 in (8.2) is $O(1/\epsilon^2)$.

For a given $\delta \in (0, 1/2]$, denote by $\Phi_{n,\delta}$ the basis defined via (8.1) by the functions

$$f_1 \equiv \frac{1}{2}, \qquad f_2(x) = \begin{cases} \frac{1}{2}\frac{1}{\sqrt{\gamma z}}, & x \in [0, \gamma]; \\ 0, & x \in (\gamma, 1]. \end{cases} \qquad \gamma := 1 - \delta^2. \tag{8.5}$$

Then

$$e(f_1) = (1-\gamma)\frac{1}{2} = \frac{\delta^2}{2}, \qquad e(f_2) = (1-\gamma)\frac{1}{2}\frac{1}{\sqrt{\gamma z}} = \frac{\delta^2}{2}\frac{1}{\sqrt{\gamma z}}$$

and by (8.2), since $\gamma < 1$

$$d_1(\Phi_{n\delta}) = \frac{1+z}{e(f_1) + z e(f_2)} = \frac{2}{\delta^2}\frac{1+z}{1+\sqrt{z}/\sqrt{\gamma}} < \frac{2}{\delta^2}\sqrt{z}. \tag{8.6}$$

With f_1, f_2 from (8.5), inequality (8.3) turns to be

$$z^\epsilon + z^{-\epsilon} > 2/\sqrt{\gamma},$$

or

$$z^\epsilon > z^{\epsilon_0} := \frac{1}{\sqrt{\gamma}} + \sqrt{\frac{1}{\gamma} - 1} = \frac{1+\sqrt{1-\gamma}}{\sqrt{\gamma}} := \sqrt{\frac{1+\delta}{1-\delta}}.$$

This implies

$$\epsilon_0 \ln z = \frac{1}{2}\ln\frac{1+\delta}{1-\delta} < \frac{1}{2}\frac{2\delta}{1-\delta} = \frac{\delta}{1-\delta} \le 2\delta,$$

i.e.,

$$\epsilon_0 < \frac{2\delta}{\ln z}.$$

Set

$$d := \frac{2}{\delta^2}\sqrt{z},$$

so that, by (8.6), we have $\Phi_{n,\delta} \in \Phi_{2,d}(\Delta_{n,z})$. Then

$$\delta = \frac{\sqrt{2} z^{1/4}}{\sqrt{d}}$$

and respectively

$$\epsilon_0 < \frac{1}{\sqrt{d}}\frac{2\sqrt{2} z^{1/4}}{\ln z}.$$

Set $z = 4$, then with $\delta \in (0, 1/2]$ we have

$$d := \frac{2}{\delta^2}\sqrt{z} \ge 16,$$

and

$$\epsilon_0 < \frac{1}{\sqrt{d}}\frac{2}{\ln 2} < \frac{3}{\sqrt{d}}.$$

9. Proof of (8.2)-(8.3)

Let $f_1, f_2 \in L_\infty[0,1]$ be given, and for

$$\Delta = \{a = t_1 < t_2 < \ldots < t_{n+1} < t_{n+2} = b\}, \quad h_i := t_{i+1} - t_i,$$

a f.e. basis $\Phi = \{\phi_i\}_{i=1}^n$ is defined as

$$\phi_i(x) = \begin{cases} f_1\left((x - t_i)/h_i\right), & x \in I_i; \\ f_2\left((x - t_{i+1})/h_{i+1}\right), & x \in I_{i+1}. \end{cases}$$

Lemma 9.1. *If $h_{i+1}/h_i = z$, then the local L_1-condition of Φ is determined by*

$$d_1(\Phi) = \frac{1+z}{e(f_1) + z e(f_2)}, \tag{8.2}$$

where

$$e(f_1) := \inf_\alpha \|f_1 - \alpha f_2\|_{L_1[0,1]}, \qquad e(f_2) := \inf_\beta \|f_2 - \beta f_1\|_{L_1[0,1]}.$$

Proof. For the value $d_1(\Phi)$ we have

$$\begin{aligned} e_i &:= \inf_{a_j} \|\phi_i - \textstyle\sum_{j \neq i} a_j \phi_j\|_{L_1(I_i \cup I_{i+1})} \\ &= \inf_{\alpha,\beta} \|\phi_i - (\alpha\phi_{i-1} + \beta\phi_{i+1})\|_{L_1(I_i \cup I_{i+1})} \\ &= \inf_\alpha \|\phi_i - \alpha\phi_{i-1}\|_{L_1(I_i)} + \inf_\beta \|\phi_i - \beta\phi_{i+1}\|_{L_1(I_{i+1})} \\ &= h_i e(f_1) + h_{i+1} e(f_2), \end{aligned}$$

where

$$e(f_1) := \inf_\alpha \|f_1 - \alpha f_2\|_{L_1[0,1]}, \qquad e(f_2) := \inf_\beta \|f_2 - \beta f_1\|_{L_1[0,1]}.$$

This gives

$$\begin{aligned} d_1(\Phi) &:= \left(\inf_i |E_i|^{-1} e_i\right)^{-1} = \sup_i \frac{h_i + h_{i+1}}{h_i e(f_1) + h_{i+1} e(f_2)} \\ &= \frac{1+z}{e(f_1) + z e(f_2)}, \end{aligned}$$

what proves (8.2).

Lemma 9.2. *If for $\epsilon \in (0, 1/2]$, and $z > 1$ holds*

$$\left(z^\epsilon + z^{-\epsilon}\right) z^{1/2} \int f_1 f_2 > \int f_1^2 + z \int f_2^2, \tag{8.3}$$

then for the inverse of the matrix

$$A_p := \{(\phi_{i,q}, \phi_{j,p})\}_{i,j=1}^n, \quad 1/p = 1/2 - \epsilon,$$

with some constants c_ϵ and $\xi_\epsilon > 1$

$$\|A_p^{-1}\|_p > |b_{1,n,p}| > c\,\xi^n. \tag{8.4}$$

Proof. For the entries of the matrix

$$A_2 := (a_{ij}) := (\phi_{i,2}, \phi_{j,2})$$

we have

$$\begin{aligned} a_{ii,2} &:= (\phi_{i,2}, \phi_{i,2}) = \frac{h_i \int f_1^2 + h_{i+1} \int f_2^2}{h_i + h_{i+1}}, \\ a_{i,i+1,2} &:= (\phi_{i,2}, \phi_{i+1,2}) = \frac{h_{i+1} \int f_1 f_2}{(h_i + h_{i+1})^{1/2} (h_{i+1} + h_{i+2})^{1/2}}, \\ a_{i,i-1,2} &:= (\phi_{i,2}, \phi_{i-1,2}) = \frac{h_i \int f_1 f_2}{(h_{i-1} + h_i)^{1/2} (h_i + h_{i+1})^{1/2}}. \end{aligned}$$

a) Consider the case of geometric mesh with the local mesh ratio z, i.e.,

$$h_{i+1}/h_i = z, \quad \forall i = \overline{1, n+1}.$$

Then

$$\begin{aligned} a_{ii,2} &= (1+z)^{-1} \left(\int f_1^2 + z \int f_2^2 \right), \\ a_{ij,2} &= (1+z)^{-1} z^{1/2} \int f_1 f_2, \quad |i-j| = 1. \end{aligned}$$

For the entries of $A_p = (a_{ij,p})$ we have

$$a_{ij,p} = (|E_j|/|E_i|)^{1/2 - 1/p} a_{ij,2}.$$

In the case of geometric mesh this gives

$$\begin{aligned} a_{ii,p} &= (1+z)^{-1} \left(\int f_1^2 + z \int f_2^2 \right), \\ a_{i,i\pm 1,p} &= (1+z)^{-1} z^{1/2 \pm \epsilon} \int f_1 f_2, \qquad \epsilon = 1/2 - 1/p, \end{aligned}$$

or

$$A_p = (1+z)^{-1} \begin{pmatrix} u & z^{\epsilon} w & & & \\ z^{-\epsilon} w & u & z^{\epsilon} w & & \\ & \ddots & \ddots & \ddots & \\ & & z^{-\epsilon} w & u & z^{\epsilon} w \\ & & & z^{-\epsilon} w & u \end{pmatrix},$$

with

$$u = \int f_1^2 + z \int f_2^2, \qquad w = z^{1/2} \int f_1 f_2. \tag{9.1}$$

b) The matrix A_p is tri-diagonal. By Cramer's rule, the element $b_{1,n,p}$ of its inverse is equal to

$$b_{1,n,p} = \frac{\det A_{n,1,p}}{\det A_p},$$

where $A_{n,1,p}$ is algebraic adjoint to $a_{n,1,p}$. It is clear that

$$\det A_{n,1,p} = \prod_{i=1}^{n-1} a_{i,i+1,p} = (1+z)^{-n+1} \alpha^{n-1}, \quad \alpha := z^{\epsilon} w.$$

As to $\det A_p$, we have

$$|A_{p,n}| = u\,|A_{p,n-1}| - w^2\,|A_{p,n-2}|,$$

whence,

$$\det A_p = (1+z)^{-n}(c_1\beta_1^n + c_2\beta_2^n),$$

with

$$\beta_{1,2} = \frac{1}{2}(u \pm \sqrt{u^2 - 4w^2}),$$

the roots of $p(t) := t^2 - ut + w^2$, and c_1, c_2 constants depending on (u, w).

Thus,

$$|b_{1,n,p}| > c\left(\frac{\alpha}{\beta_1}\right)^n, \quad c = c(u,w,z),$$

and respectively

$$\|A_p^{-1}\|_p > |b_{1,n,p}| > c\,\xi^n, \quad \xi = \frac{\alpha}{\beta_1}, \qquad 1/p = 1/2 - \epsilon.$$

c) We obtain $\xi > 1$, as required in (8.4), if $\alpha > \beta_1$. This is the inequality

$$z^\epsilon w > \frac{1}{2}(u + \sqrt{u^2 - 4w^2}),$$

or

$$2z^\epsilon w - u > 0, \qquad (2z^\epsilon w - u)^2 > u^2 - 4w^2,$$

i.e.,

$$(z^\epsilon + z^{-\epsilon})w > u, \qquad z > 1$$

Substituting expressions for u, w from (9.1) we obtain (8.3):

$$(z^\epsilon + z^{-\epsilon})\,z^{1/2}\int f_1 f_2 > \int f_1^2 + z\int f_2^2.$$

10. Appendix: A proof of de Boor's formula

Lemma 10.2. *Let $p \in [1,\infty]$, $1/p + 1/q = 1$, and let $S \in L_p$ be a linear subspace. Then*

$$\|P_S\|_p = \sup_{s\in S}\inf_{\sigma\in S}\frac{\|s\|_p\|\sigma\|_q}{|(s,\sigma)|} = \sup_{\sigma\in S}\inf_{s\in S}\frac{\|s\|_p\|\sigma\|_q}{|(s,\sigma)|} \tag{10.1}$$

Proof. We use

(a) the observation that

$$\sup_{s\in S}\inf_{\sigma\in S}\frac{\|s\|_p\|\sigma\|_q}{|(s,\sigma)|} = \sup_{Pf\in S}\inf_{Pg\in S}\frac{\|Pf\|_p\|Pg\|_q}{|(Pf,Pg)|} = \sup_{f\in L_p}\inf_{g\in L_q}\frac{\|Pf\|_p\|Pg\|_q}{|(Pf,Pg)|},$$

(b) the relations

$$(Pf, g) = (Pf, Pg) = (f, Pg), \qquad \|Pg\|_q \le \|P\|_q\|g\|_q$$

which are due to the definition of the orthoprojector P, and

(c) Hölder (in)equality in the form

$$\inf_{v \in L_q} \frac{||u||_p ||v||_q}{|(u,v)|} = 1.$$

Now we have

$$\begin{aligned}
\sup_{f \in L_p} \inf_{g \in L_q} \frac{||Pf||_p ||Pg||_q}{|(Pf,Pg)|} &= \sup_{f \in L_p} \inf_{g \in L_q} \frac{||Pf||_p ||Pg||_q}{|(Pf,g)|} \\
&\leq ||P||_q \sup_{f \in L_p} \inf_{g \in L_q} \frac{||Pf||_p ||g||_q}{|(Pf,g)|} \\
&= ||P||_q,
\end{aligned}$$

$$\begin{aligned}
\sup_{f \in L_p} \inf_{g \in L_q} \frac{||Pf||_p ||Pg||_q}{|(Pf,Pg)|} &= \sup_{f \in L_p} \inf_{g \in L_q} \frac{||Pf||_p ||Pg||_q}{|(f,Pg)|} \\
&\geq \sup_{f \in L_p} \inf_{g \in L_q} \frac{||Pf||_p ||Pg||_q}{||f||p||Pg||_q} = \sup_{f \in L_p} \frac{||Pf||_p}{||f||_p} \\
&= ||P||_p,
\end{aligned}$$

i.e.,

$$||P_S||_p \leq \sup_{s \in S} \inf_{\sigma \in S} \frac{||s||_p ||\sigma||_q}{|(s,\sigma)|} \leq ||P_S||_q.$$

Similarly,

$$||P_S||_q \leq \sup_{\sigma \in S} \inf_{s \in S} \frac{||s||_p ||\sigma||_q}{|(s,\sigma)|} \leq ||P_S||_p,$$

what completes the proof.