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On a Problem of C. de Boor for Multivariate D^m-Splines

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1. INTRODUCTION AND STATEMENT OF THE PROBLEM

One of the ways of defining multivariate splines is a variational approach leading to Atiyah's D^m-splines:

$$s = s(f, m, \Delta, \Omega) = \arg\ \min\{\, \|D^m g\|_2 :\ g \in W_2^m(\Omega),\ g|_\Delta = f|_\Delta \}\,. \tag{1.1}$$

Here $\Omega \subset R^n$ is a bounded domain with a smooth boundary and Δ is a closed subset of Ω,

$$\|D^l g\|_p = \begin{cases} \left\{ \int\limits_\Omega \left(\sum\limits_{|\alpha|=l} \dfrac{l!}{\alpha!} \, |D^\alpha g|^2 \right)^{p/2} dx \right\}^{1/p}, & 1 \le p < \infty, \\ \max\limits_{|\alpha|=l} \|D^\alpha g\|_{L_\infty(\Omega)}, & p = \infty. \end{cases}$$

For $n = 1$, the D^m-splines are ordinary piecewise-polynomial functions of degree $2m-1$. For $n > 1$, the D^m-splines are polyharmonic in the domain $\Omega \setminus \Delta$ with order m, i.e., $\nabla^{2m} s \equiv 0$, where ∇^2 is a Laplace operator.

The D^m-splines inherit a number of important properties of one-dimensional splines of degree $2m - 1$. In particular, we denote by

$$\overline{h}_\nu = \sup_{x\in\Omega} \inf_{y\in\Delta} |x - y|, \quad \underline{h}_\nu = \inf_{y,z\in\Delta} |y - z|$$

the maximal and the minimal step of the net $\Delta = \Delta_\nu$. Then the condition $f \in W_2^m(\Omega)$ ensures the convergence

$$\|f - s_\nu(f)\|_{W_2^m(\Omega)} \to 0\,, \quad \overline{h}_\nu \to 0$$

irrespective of the technique of condensation of the nets Δ_ν, in particular, irrespective of the boundedness of the quantity

$$M_\nu = \overline{h}_\nu / \underline{h}_\nu.$$

We shall call the convergence of this kind an *unconditional convergence* of spline-interpolants.

In Sobolev spaces $W_p^l(\Omega)$ which are different from $W_2^m(\Omega)$ the convergence of D^m-splines can be ensured on quasiuniform nets Δ_ν, i.e., under the condition

$$M_\nu < M, \quad \nu \in \mathbb{N}.$$

However, when there are no constraints, the examples of divergence of an interpolation process in some $W_p^l(\Omega)$ do exist (the details are given below).

Definition 1. We say that for the given l, p, m, n, Ω an unconditional convergence (un. con.) of D^m-splines takes place in the space $W_p^l(\Omega)$ and write

$$s_\nu(f) \to f \quad \text{uncond. in } W_p^l(\Omega)$$

if, for any function $f \in W_p^l(\Omega)$, any sequence of discrete nets Δ_ν, and any compact set $B \subset \Omega$ we have the convergence

$$\|f - s_\nu(f)\|_{W_p^l(B)} \to 0\,, \quad \overline{h}_\nu \to 0.$$

In this work we study the following problem.

Problem. Being given m, n, Ω, find the necessary and sufficient conditions imposed on $l \in \mathbb{N}$ and $p \in [1, \infty]$ for which

$$s_\nu(f) \to f \quad \text{uncond. in } W_p^l(\Omega),$$

This problem was posed by Yu.N. Subbotin. Its origination is connected with the well-known conjecture of C. de Boor [1] for one-dimensional splines, namely, *for any function $f \in C^m[a, b]$ and any discrete net $\Delta_\nu \subset [a, b]$ the estimate*

$$\|s_\nu^{(m)}(f)\|_\infty \le c_m \|f^{(m)}\|_\infty$$

with the constant c_m independent of Δ_ν holds true.

2. THE WELL-POSEDNESS OF THE PROBLEM FOR DISCRETE NETS

The following conditions are necessary for the problem to be well-posed:

(A) the existence and uniqueness of the D^m-spline $s_\nu(f)$ determined in (1.1) from the values of the function $f \in W_p^l(\Omega)$ on Δ_ν;

(B) the inclusion $s_\nu(f) \in W_p^l(\Omega, \mathrm{loc})$.

Our considerations refer to the case of discrete nets

$$\Delta_\nu = \{t_{i\nu}\}_{i=1}^{N_\nu}, \quad \underline{h}_\nu > 0.$$

In this case, conditions (A), (B) will be fulfilled under the following assumptions:

(a) $m > n/2$, $l > n/p$;

(b) $l - n/p < 2m - n$.

Indeed, condition (a) entails the embeddings

$$W_2^m(\Omega) \to C(\Omega), \quad W_p^l(\Omega) \to C(\Omega),$$

and this gives (A).

Condition (b) ensures the inclusion $G \in W_p^l(R^n, \mathrm{loc})$ for the function

$$G(x) = \begin{cases} |x|^{2m-n}, & n = 2n_1; \\ |x|^{2m-n} \ln |x|, & n = 2n_1 + 1, \end{cases}$$

which is a fundamental solution of the polyharmonic equation $\nabla^{2m} u = g$. This is equivalent to condition (B) since [6] the D^m-spline on the discrete net $\Delta = \{t_i\}_{i=1}^N$ can be represented as

$$s(x) = \sum_{i=1}^{N} c_i G(x - t_i) + F(x)$$

with the function $F(x) = F(x; m, \Delta, \Omega)$ which is polyharmonic and, consequently, analytic in the domain Ω.

It is easy to note that conditions (a), (*b*) can be combined into one condition

$$0 < l - n/p < 2m - n,$$

which we shall assume to be fulfilled in the sequel, drawing attention to it only from time to time.

3. HISTORY OF THE PROBLEM

3.1. One-dimensional case. The first example of the divergence of an interpolation process belongs to Nord [9], namely, an example of the divergence of cubic splines ($m = 2$) in $C[a, b]$. From subsequent works only very general results should be noted [5, 2]. A simple technique of construction of these examples was proposed in [7]. The available results constitute the following statement.

Theorem A1 (necessary condition for uncond. conv.). *Let*

$$s_\nu(f) \to f \quad \textit{uncond. in } W_p^l[a, b].$$

Then one of the following conditions is fulfilled:

$$\begin{array}{llll} (1) & l = m, & p \in [1, \infty], & n = 1; \\ (2) & l = m + 1, & p = 1, & n = 1; \\ (3) & l = m - 1, & p = \infty, & n = 1. \end{array}$$

C. de Boor's conjecture [1, 2] consists in the assertion that for $n = 1$ *conditions* (1)–(3) *of Theorem* A1 *are necessary and sufficient for the unconditional convergence of the splines in* $W_p^l[a, b]$.

The following theorem provides a partial justification of this conjecture.

Theorem A2 (sufficient condition for uncond. conv.). *Suppose that one of the following conditions is fulfilled:*

$$\begin{array}{lllll} (1a) & l = m, & p \in [1, \infty], & m = 2, 3, & n = 1; \\ (1b) & l = m, & p \in (2 - \epsilon_m, 2 + \epsilon'_m) & m \geq 4, & n = 1; \\ (2) & l = m + 1, & p = 1, & m = 2, & n = 1; \\ (3) & l = m - 1, & p = \infty, & m = 2, & n = 1. \end{array}$$

Then

$$s_\nu(f) \to f \quad \textit{uncond. in } W_p^l[a,b].$$

The sufficiency of conditions (1a), (2), (3), which refer to small values of m, was proved in [11, 3, 4]. Condition (1b), which ensures the unconditional convergence of the splines in $W_p^m[a,b]$ for any m, provided that p is sufficiently close to 2, is our recent result [10].

3.2. Multivariate case. For $n > 1$ the questions concerning the approximation of the function $f \in W_p^l(\Omega)$ by means of D^m-spline interpolants were considered by Matveev [7, 8]. As concerns de Boor's problem, he got the following results.

Lemma B [8]. *Let $n \geq 1$, $m > n/2$, and let $I^n = (-1,1)^n$ be an n-dimensional cube,*

$$\mathcal{A}_n^m := \left\{(l,p) : \ s_\nu(m,f) \to f \quad \textit{uncond. in } W_p^l(I^n)\right\}.$$

Then

$$\mathcal{A}_1^m \supseteq \mathcal{A}_2^m \supseteq \ldots \supseteq \mathcal{A}_n^m \supseteq \ldots$$

In other words, with an increase in the dimension, the set of Sobolev spaces $W_p^l(\Omega)$ which admit the unconditional convergence of D^m-splines is not considered, to say the least. In particular, the necessary conditions presented in Theorem A1 for $n = 1$ are automatically generalized to spaces of dimension $n > 1$.

Matveev also announced the result [7] which shows that, actually, with an increase in the dimension n the set of spaces $W_p^l(\Omega)$ which admit the unconditional convergence of D^m-splines narrows.

Theorem C1 (necessary condition for uncond. conv.). *Let*

$$s_\nu(f) \to f \quad \textit{uncond. in } W_p^l(\Omega).$$

Then one of the following conditions is fulfilled:

$$\begin{array}{llll} (1a) & l = m, & p \in [1,\infty), & n = 2; \\ (1b) & l = m, & p \in \left[2 - [\frac{n+1}{2}]^{-1}, 2 + [\frac{n-1}{2}]^{-1}\right], & n \geq 3; \\ (2) & l = m+1, & p = 1, & n = 2,3,4; \\ (3) & l = m-1, & p = \infty, & n = 2,3. \end{array} \tag{3.1}$$

At the same time, the following sufficient condition for unconditional convergence is well-known.

Theorem C2 (sufficient condition for uncond. conv.).

$$s_\nu(f) \to f \quad \textit{uncond. in } W_2^m(\Omega).$$

It is obvious that as $n \to \infty$, the necessary conditions (3.1) for the unconditional convergence $s_\nu(f,m) \to f$ in $W_p^l(\Omega)$ are asymptotically close to the sufficient condition

$$l = m, \quad p = 2. \tag{3.2}$$

Here, in contrast to the one-dimensional case, no other sufficient conditions are known.

In this work, we show that this fact is not accidental and that, with a small exception, condition (3.2) is the only necessary and sufficient condition for unconditional convergence $s_\nu(f,m) \to f$ in $W_p^l(\Omega)$.

4. FORMULATION OF THE RESULT

4.1. The main result. We have proved the following theorem.

Theorem 1 (necessary condition for uncond. conv.). *Let l, m, n, p satisfy the inequalities $0 < l - n/p < 2m - n$ and be such that*

$$s_\nu(f) \to f \quad \textit{uncond. in } W_p^l(I^n).$$

Then either

$$(l,p) = (m,2), \quad n \geq 2, \tag{4.1}$$

or one of the following conditions is fulfilled:

$$\begin{array}{llllll} (1a) & l = m, & p \in [1,2), & m = 2m_1, & n = 2; \\ (1b) & l = m, & p \in [3/2,2), & m = 2m_1, & n = 3; \\ (2a) & l = m+1, & p = 1, & & n = 2,3; \\ (2b) & l = m+1, & p = 1, & m = 2m_1+1, & n = 4; \\ (3) & l = m-1, & p = \infty, & & n = 2,3. \end{array} \tag{4.2}$$

Corollary. *For $n \geq 5$, the convergence*

$$s_\nu(f,m) \to f \quad \textit{uncond. in } W_p^l(I^n)$$

takes place if and only if

$$(l,p) = (m,2).$$

Due to our results, it should be expected that, actually, the following conjecture is valid.

Conjecture 1. *For $n \geq 2$ the convergence*

$$s_\nu(f,m) \to f \quad \textit{uncond. in } W_p^l(I^n)$$

takes place if and only if

$$(l,p) = (m,2).$$

In order to get this final result, the counterexamples are lacking only in three cases for $n = 2$ (see Remark 4.1 below).

4.2. Reduction to a small dimension n. We have to show that if none of conditions (4.1), (4.2) is fulfilled, then there exist $f \in W_p^l(I^n)$, $B \in I^n$, and a sequence $\{\Delta_\nu\}$ for which the norms $\|s_\nu(f)\|_{W_p^l(B)}$ will increase indefinitely.

According to the Banach–Steinhaus theorem, by virtue of Lemma B, this is a direct corollary of the following result.

Theorem 1′. *Suppose that one of the following conditions is fulfilled:*

$$
\begin{array}{llllll}
\text{(1a)} & l=m, & p\in(2,\infty], & & n=2; & \\
\text{(1b)} & l=m, & p\in[1,2), & m=2m_1+1, & n=2; & \\
\text{(1c)} & l=m, & p\in[1,3/2), & m=2m_1+1, & n=3; & \\
\text{(1d)} & l=m, & p\in[1,2), & m=2m_1, & n=4; & \qquad (4.3)\\
\text{(2a)} & l=m+1, & p=1, & m=2m_1, & n=4; & \\
\text{(2b)} & l=m+1, & p=1, & m=2m_1+1, & n=5; & \\
\text{(3)} & l=m-1, & p=\infty, & & n=4. &
\end{array}
$$

Then, for any $M,\epsilon,\eta>0$, there exist a function $f\in W_p^l(I^n)$ and a discrete net Δ_ν such that

$$\operatorname{dist}(\Delta_\nu, I^n)<\epsilon,\quad \|f\|_{W_p^l(I^n)}=1,\quad \|s(f,m,\Delta_\nu,I^n)\|_{W_p^l(\eta I^n)}>M. \tag{4.4}$$

Indeed, according to Lemma B, Theorem 1′ is valid for all n beginning with those indicated in (4.3). And from this theorem, according to the Banach–Steinhaus theorem, follows the existence of a function $g\in W_p^l(I^n)$ and a sequence of nets Δ_ν such that

$$\|s(g,m,\Delta_\nu,I^n)\|_{W_p^l(\eta I^n)}\to\infty,\quad \overline{h}_\nu\to 0,$$

for the given $\eta>0$

Remark 4.1. We can see from (4.3) that in order to prove the hypothesis concerning the uniqueness of the sufficient condition for the unconditional convergence in $W_p^l(I^n)$

$$(l,p)=(m,2),\quad n\ge 2,$$

it suffices to construct examples similar to (4.4) only in the following three cases:

$$
\begin{array}{lllll}
\text{(i)} & l=m, & p\in[1,2), & m=2m_1, & n=2;\\
\text{(ii)} & l=m+1, & p=1, & & n=2;\\
\text{(iii)} & l=m-1, & p=\infty, & & n=2.
\end{array}
$$

4.3. The structure of the work. The rest part of the article is connected with the proof of Theorem 1′. The case $l=m,m+1$ is discussed in Secs. 5–16 and the case $l=m-1$ in Sec. 17. In Secs. 5, 6 and at the beginning of Sec. 17 we carry out further simplifications which reduce Theorem 1′ to Theorem 2 ($l=m,m+1$) and Theorem 3 ($l=m-1$).

Theorem 2 ($l=m,m+1$) is formulated in Sec. 7 and each of its cases is then proved in Secs. 10–16. Since the counterexamples for $l=m,m+1$ are very complicated, we give detailed proofs only in cases (1a), (1b) where $n=2$. In the other cases ($n=3$, 4, 5), we restrict ourselves to the formulation of statements and brief explanations.

Theorem 3 and everything that refers to the case $l=m-1$ can be found in Sec. 17. Here we also restrict ourselves to a brief exposition.

5. REDUCTION TO INTERPOLATION ON AN ARBITRARY SET

In this section we show the possibility of a further simplification in cases (1), (2) of Theorem 1′ where $l = m, m+1$, namely, we show that in these cases the existence of $f \in W_p^l(I^n)$ and of a discrete net Δ_ν with properties (4.4) follows from the existence of $f \in W_p^l(I^n) \cap W_2^m(I^n)$ and an already arbitrary (not necessarily discrete) set Δ with the same properties (4.4).

Lemma 5.1. *Let $m > n/2$, $\Delta \subset I^n$ and let the sequence of discrete nets $\{\Delta_i\}$ be such that*

$$\Delta_i \subset \Delta_{i+1} \subset \Delta, \quad \operatorname{dist}(\Delta_i, \Delta) \to 0.$$

Suppose, furthermore, that

$$s_i := s(f, m, \Delta_i, I^n), \quad s := s(f, m, \Delta, I^n)$$

for the given $f \in W_2^m(I^n)$.

Then

$$\|s_i - s\|_{W_2^m(I^n)} \to 0.$$

The proof is similar to that given in [7, p. 150].

Remark 5.1. If we choose as Δ the closure of a certain subdomain I^n with a sufficiently smooth boundary S, then, by virtue of the smoothness of $s, f \in W_2^m(I^n)$, the interpolation $s(f)\Big|_\Delta \equiv f\Big|_\Delta$ entailes the interpolation of the boundary values of f, i.e.,

$$\frac{\partial^k s(f)}{\partial n_S^k}\Big|_S = \frac{\partial^k f}{\partial n_S^k}\Big|_S, \quad k = 0, \ldots, m-1,$$

where n_S is a normal vector to S.

Let us now consider the problem concerning the estimation of the W_p^l-norms of $s_i(f)$ in terms of the W_p^l-norm of the limit spline $s(f, m, \Delta, I^n)$.

Generally speaking, for a nondiscrete Δ, for any smoothness of f, we can a priori state the only fact that $s(f) \in W_2^m(I^n)$. However, since the function $s(f)$ is polyharmonic and, consequently, analitic in $V = (I^n \setminus \Delta)$, we have the inclusion $s(f) \in W_p^l(B)$ for any l, p and any compact set $B \in V$.

The following lemma shows that on any compact set of this kind the W_p^l-norms of the discrete splines s_i also converge to $\|s\|_{W_p^l(B)}$.

Lemma 5.2. *Let $m > n/2$, $f \in W_2^m(I^n)$, $\Delta_i, \Delta \subset I^n$,*

$$\|s - s_i\|_{W_2^m(I^n)} \to 0.$$

Then, for any compact set B such that

$$B \subset V := I^n \setminus (\{\Delta_i\}_1^\infty \cup \Delta),$$

for any l, p, the convergence

$$\|s - s_i\|_{W_p^l(B)} \to 0$$

takes place.

Proof. For the functions g which are polyharmonic in the ball $B(a, 2\epsilon)$ we have an inequality of the Markov's type, namely,

$$\|g\|_{W_p^l[B(a,\epsilon)]} \le c(\epsilon, l, p)\|g\|_{L_1[B(a,2\epsilon)]}. \tag{5.1}$$

Furthermore, for any small $\epsilon > 0$ there exists a covering B by a finite number of balls $B(a_j, \epsilon)$ of radius ϵ, say, by the number of balls $K = K(B, \epsilon)$, such that

$$B \subset \cup_1^K B(a_j, \epsilon) \subset \cup_1^K B(a_j, 2\epsilon) \subset (I^n \setminus \Delta).$$

Since the D^m-splines s, s_i are polyharmonic in the domain V that contains B, we can set $f_i = s - s_i$ and apply (5.1) to obtain

$$\|f_i\|_{W_p^l(B)} \le \sum_{j=1}^K \|f_i\|_{W_p^l[B(a_j,\epsilon)]} \le c(\epsilon, l, p) \sum_{j=1}^K \|f_i\|_{L_1[B(a_j,2\epsilon)]} \le K(B, \epsilon)c(\epsilon, l, p)\|f_i\|_{L_1(I^n)},$$

i.e.,

$$\|s - s_i\|_{W_p^l(B)} \le c_1(B, \epsilon, l, p)\|s - s_i\|_{L_1(I^n)}.$$

However,

$$\|s - s_i\|_{L_1(I^n)} \le \|s - s_i\|_{W_2^m(I^n)} \to 0,$$

and this completes the proof of the lemma.

Lemmas 5.1, 5.2 allow us to reduce Theorem $1'$ to splines on arbitrary sets $\Delta \in I^n$. However, we shall not do this now, in the general situation, but shall formulate the corresponding statement in Sec. 7 for special $f, s(f), \Delta$ defined in Sec. 6.

6. SPECIAL ELEMENTS OF CONSTRUCTIONS

Everywhere in what follows we have

$$x = (x_1, \ldots, x_n), \quad r = |x| = \sqrt{x_1^2 + \ldots + x_n^2}.$$

6.1. The choice of the net Δ. For the arbitrary $0 < h < H < 2^n$ we set

$$V := V_{H,h} := \{x \in I^n : \ h < r < H\}, \quad U := U_h := \{x \in I^n : \ 0 \le r \le h\}.$$

As the set Δ on which the condition of interpolation $s|_\Delta = f|_\Delta$ is defined we set

$$\Delta := \Delta_{H,h} := I^n \setminus V.$$

6.2. The choice of the function f. For every collection (m, n, l, p) the function f is radial, i.e.,

$$f(x) = f(r).$$

6.3. The explicit form of D^m-splines. For the interpolation D^m-spline $s = s(f,m,\Delta_{H,h},I^n)$ we have

$$s(x) \in W_2^m(I^n), \qquad s(x) \equiv f(x), \quad x \in \Delta_{H,h}, \tag{6.1}$$

and also

$$\nabla^{2m} s = 0, \quad x \in V = (I^n \setminus \Delta_{H,h}). \tag{6.2}$$

Furthermore, since f also belongs to $W_2^m(I^n)$, conditions (6.1) imply

$$\frac{\partial^k s}{\partial r^k}\Big|_{r=H,h} = \frac{\partial^k f}{\partial r^k}\Big|_{r=H,h}, \quad k = 0,\ \dots, m-1. \tag{6.3}$$

By virtue of the existence and uniqueness of the solution of the Dirichlet problem for a polyharmonic equation, the function σ which satisfies conditions (6.2), (6.3), is a restriction of the D^m-spline $s(f)$ to the ring $V_{H,h}$, i.e.,

$$\sigma(f,x) \equiv s(f,x), \quad x \in V_{H,h}.$$

We shall also seek $\sigma(f,x)$ in the form of the radial function $\sigma(r)$. The functions

$$\begin{aligned}
n &= 2, \qquad \sigma(r) = \textstyle\sum_{j=1}^m a_j r^{2j-2} + \sum_{j=1}^m b_j r^{2j-2} \ln r,\\
n &= 3, \qquad \sigma(r) = \textstyle\sum_{j=1}^m a_j r^{2j-2} + \sum_{j=1}^m b_j r^{2j-3},\\
n &= 4, \qquad \sigma(r) = \textstyle\sum_{j=1}^m a_j r^{2j-2} + b_1/r^2 + \sum_{j=2}^m b_j r^{2j-4} \ln r,\\
n &= 5, \qquad \sigma(r) = \textstyle\sum_{j=1}^m a_j r^{2j-2} + \sum_{j=1}^m b_j r^{2j-5}
\end{aligned} \tag{6.4}$$

are radial functions which are m-harmonic in $\mathbb{R}^n \setminus \{0\}$. In this case, the jth terms of every sum are polyharmonic, with the exact order j.

The boundary conditions (6.3) define $\sigma(r)$ of the form (6.4) also uniquely. Thus, for our choice of $\Delta = \Delta_{H,h}$ and $f = f(r)$, we have

$$s(f,m,\Delta_{H,h},I^n;x) = \sigma(f,r), \quad x \in V_{H,h},$$

where $\sigma(f,r)$ has the form (6.4) and satisfies the boundary conditions (6.3).

6.4. Inequalities for W_p^l-norms. We set

$$\|g\|_{W_p^l(I^n)} := \|g\|_{L_p(I^n)} + \|D^l g\|_{L_p(I^n)},$$

where

$$\|D^l g\|_{L_p(I^n)} = \begin{cases} \left\{ \displaystyle\int_{I^n} \left(\sum_{|\alpha|=l} \frac{l!}{\alpha!} |D^\alpha g(x)|^2 \right)^{p/2} dx \right\}^{1/p}, & 1 \le p < \infty; \\ \displaystyle\max_{|\alpha|=l} \|D^\alpha g\|_{L_\infty(I^n)}, & p = \infty. \end{cases}$$

In this case, as usual, $\alpha = (\alpha_1,\ \dots, \alpha_n)$ is a multi-index,

$$|\alpha| = \alpha_1 + \ldots + \alpha_n, \quad \alpha! = \alpha_1! \ldots \alpha_n!, \quad D^\alpha g(x) = \frac{\partial^{|\alpha|} g(x)}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

Let us find, for these norms, the lower and upper estimates that will be convenient for the consideration of the radial functions $g(x) = g(r)$.

1. To find the lower estimate, we use the relation

$$\frac{\partial^l g(r)}{\partial r^l} = \sum_{|\alpha|=l} \frac{l!}{\alpha!} \frac{x^\alpha}{r^l} D^\alpha g(x).$$

Applying the Schwartz inequality to the right-hand side, we get

$$\left|\frac{\partial^l g(r)}{\partial r^l}\right|^2 \le \left(\sum_{|\alpha|=l} \frac{l!}{\alpha!} \frac{x^{2\alpha}}{r^{2l}}\right) \left(\sum_{|\alpha|=l} \frac{l!}{\alpha!} |D^\alpha g(x)|^2\right) = \sum_{|\alpha|=l} \frac{l!}{\alpha!} |D^\alpha g(x)|^2,$$

whence it follows that

$$\|D^l g\|_{L_p(V)} \ge c_n \left\{ \int_h^H \left|\frac{\partial^l g(r)}{\partial r^l}\right| r^{n-1}\, dr \right\}^{1/p}. \tag{6.5}$$

2. In order to find the upper estimate, we use the relation

$$D^\alpha g(x) = \sum_{k=1}^{l} \sum_{|\beta|=k} c_{\alpha\beta} \frac{x^\beta}{r^k} \frac{1}{r^{l-k}} \frac{\partial^k g(r)}{\partial r^k}.$$

From this relation, we derive

$$\|D^l g\|_{L_p(I^n)} \le c_{l,n} \max_{1\le k\le l} \left\{ \int_0^2 \left|\frac{1}{r^{l-k}} \frac{\partial^k g(r)}{\partial r^k}\right|^p r^{n-1}\, dr \right\}^{1/p}. \tag{6.6}$$

7. STATEMENT BEING PROVED

It will follow from the results of Secs. 6, 7 that in order to prove Theorem 1′ in the cases $l = m, m+1$, it suffices to prove the following statement.

Theorem 2. *Suppose that one of the following conditions is fulfilled:*

$$\begin{array}{lllll} (1a) & l = m, & p \in (2,\infty], & & n = 2; \\ (1b) & l = m, & p \in [1,2), & m = 2m_1+1, & n = 2; \\ (1c) & l = m, & p \in [1,3/2), & m = 2m_1+1, & n = 3; \\ (1d) & l = m, & p \in [1,2), & m = 2m_1, & n = 4; \\ (2a) & l = m+1, & p = 1, & m = 2m_1 & n = 4; \\ (2b) & l = m+1, & p = 1, & m = 2m_1+1, & n = 5. \end{array} \tag{7.1}$$

Then, for any $M, H > 0$, there exist $h > 0$ and a function $f = f_{H,h}$ such that

$$0 < h < H, \quad f \in W_p^l(I^n) \cap W_2^m(I^n), \quad \|f\|_{W_p^l(I^n)} = O_p(1), \tag{7.2}$$

and for the spline $\sigma(f)$ *such that*

$$\nabla^{2m}\sigma = 0, \quad x \in V; \qquad \frac{\partial^k[f-\sigma]}{\partial r^k}\Big|_{r=H,h} = 0, \quad k = 0, \ \ldots, m-1, \tag{7.3}$$

the inequality

$$\|\sigma(f)\|_{W_p^l(V)} > M \tag{7.4}$$

is satisfied.

Indeed, for the arbitrary $\epsilon, \eta > 0$ defined in Theorem 1′, we take

$$H := \min\{\epsilon, \eta\}.$$

Then we have

$$\text{dist}\,(\Delta_{H,h}, I^n) = H - h < \epsilon, \quad V = V_{H,h} \subset \eta I^n$$

for our special $\Delta_{H,h} = I^n \setminus V_{H,h}$ for the arbitrary $h < H$. With the chosen H and defined M, we choose h and $f_{H,h}$ such that relations (7.2)–(7.4) would be satisfied. In this case, the inequality

$$\|\sigma(f)\|_{W_p^l(B)} > M$$

wiil also be satisfied for a certain compact set $B \subset V$.

If we now choose a sequence of discrete nets Δ_i such that

$$\Delta_i \subset \Delta_{i+1} \subset \Delta_{H,h}, \quad \text{dist}\,(\Delta_i, \Delta_{H,h}) \to 0,$$

then, by virtue of Lemmas 5.1, 5.2,

$$\|s_i(f)\|_{W_p^l(B)} \to \|s(f)\|_{W_p^l(B)} = \|\sigma(f)\|_{W_p^l(B)}$$

and, hence, for a certain ν we shall also have

$$\text{dist}(\Delta_\nu, I^n) < \epsilon, \quad \|s_\nu(f)\|_{W_p^l(\eta I^n)} > \|s_\nu(f)\|_{W_p^l(B)} > M.$$

8. TWO EXAMPLES AND THE GENERAL IDEA

The general construction of the function f for the arbitrary m is based on the following simple observations for $n = 2$, $m = 1$.

8.1. Example for $p > 2$**.** Let $n = 2$, $m = 1$ and let $V := V_{1,h} = \{x : \ h < r < 1\}$. We take a smooth function $f = f(r)$ such that

$$f(r) = \begin{cases} 0, & r = 1, \\ 1, & r = h, \end{cases}$$

say, $f(r) = f_h(r) = (1 - r^2)/(1 - h^2)$. The solution of the problem

$$\nabla^2\sigma(x) = 0, \quad x \in V; \qquad \sigma(r)|_{r=1,h} = f_h(r)|_{r=1,h}$$

is the D^1-spline

$$\sigma_h(f_h, r) = \ln r / \ln h, \quad h < r < 1.$$

From this, using the relation $(\frac{\partial g}{\partial x_1})^2 + (\frac{\partial g}{\partial x_2})^2 = (\frac{\partial g}{\partial r})^2$, we obtain

$$\|D^1 \sigma_h\|_{L_2(V)} = \frac{1}{\sqrt{|\ln h|}} \to 0, \quad h \to 0,$$

whereas for $p > 2$, we have

$$\|D^1 \sigma_h\|_{L_p(V)} = \frac{O_p(1)}{h^{1-2/p} |\ln h|} \to \infty, \quad h \to 0.$$

Thus, even for infinitely smooth $f = f_h$ with $\|f_h\|_{W^1_\infty(I^2)} < c$, we have

$$\|\sigma_h(f_h)\|_{W^1_p(V_{1,h})} \to \infty, \quad h \to 0, \quad p > 2.$$

8.2. Example for $p < 2$. Suppose that $n = 2$, $m = 1$, $V = \{x : \ h < r < 1\}$ as before. We take

$$f(r) = \ln^2 r.$$

Then, for any $\epsilon > 0$, we have

$$f \in W^1_{2-\epsilon}(I^2), \quad \|f\|_{W^1_{2-\epsilon}(I^2)} < K_\epsilon < \infty.$$

The solution of the problem

$$\nabla^2 \sigma = 0, \quad x \in V; \qquad \sigma(r)|_{r=H,h} = f(r)|_{H,h}$$

is the D^1-spline

$$\sigma_h(f, r) = \ln r \cdot \ln h, \quad h < r < 1.$$

Since the inequality $\|\ln r\|_{L_1(V_h)} > C_0$ is satisfied for small $h < h_0$, it is obvious that not only for the W^1_p-norm but also for the L_1-norm we get

$$\|\sigma_h\|_{L_1(V_{1,h})} = O(|\ln h|) \to \infty, \quad h \to 0.$$

8.3. The idea of the general construction. 1. For $n = 2$, $m = 2m_1 + 1$, $p > 2$ we construct a smooth function $f \in W^m_\infty(I^2)$ of the form

$$f(r) = r^{2m} + \dots,$$

for which, on the ring $V = V_{H,h}$, we have

$$\sigma(f, r) = c_m(H) r^{2m_1} \ln r / \ln h + \dots,$$

so that for $p > 2$ we obtain

$$\|D^{2m_1+1} \sigma\|_{L_p(V)} \geq c_m \|1/(r \ln h)\|_{L_p(V)} \to \infty\,, \quad h \to 0,$$

just as in Example 8.1.

2. For $n = 2$, $m = 2m_1 + 1$, $p < 2$ we construct a function $f \in W_p^m(I^2)$ of the form

$$f(r) = r^{2m_1} \ln^2 r + \dots,$$

for which, on the ring $V_{H,h}$, we have

$$\sigma(f, r) = c_m(H) r^{2m_1} \ln r \cdot \ln h + \dots,$$

so that

$$\|\sigma\|_{L_1(V)} \to \infty, \quad h \to 0,$$

just as in Example 8.2.

3. The construction itself is very cumbersome. It could have been simpler if there existed a simple formula for Hermite's two-point interpolation for the system

$$\{r^{2j}, r^{2j} \ln r\}_{j=0}^{m-1}$$

(such, for instance, as the formula for Hermite's polynomial interpolation). Then we could have immediately written out the explicit form of Hermite's D^m-interpolant $\sigma(f)$, say, for

$$f(r) = r^{2m}, \quad p > 2; \qquad f(r) = r^{m-1} \ln^2 r, \quad p < 2.$$

However, we could not find formulas of this kind.

4. Therefore we begin with constructing a function F such that

$$\frac{\partial^k F(r)}{\partial r^k}\bigg|_{r=H,h} = 0, \quad k = 0, \dots, m-1,$$

and then decompose it into two parts

$$F = f - \sigma(f),$$

referring to $\sigma(f)$ polyharmonic terms of F of order not higher than m, not necessarily all terms of this kind. Thus the boundary conditions and the polyharmonicity of $\sigma(f)$ are satisfied automatically, and it remains to estimate the corresponding norms of f and $\sigma(f)$.

9. THE FUNCTIONS $\phi_m(r, h)$

The basic element of the general construction for $n = 2$ is the radial function ϕ_m, which is polyharmonic in $\mathbb{R}^2 \setminus \{0\}$, defined as follows.

We set

$$\phi_1(r, h) := \ln r - \ln h$$

and define

$$\phi_m(r, h) := \int_h^r \frac{1}{u} \int_h^u t \cdot \phi_{m-2}(t, h)\, dt\, du$$

for odd $m = 2m_1 + 1 = 3, 5, \ldots$.

Lemma 9.1. *The function* ϕ_m *satisfies the conditions*

$$\frac{\partial^k \phi_m(r,h)}{\partial r^k}\Big|_{r=h} = 0, \quad k = 0, \ldots, m-1, \tag{9.1}$$

and has the form

$$\phi_m(r,h) = p_{m-1}(r,h)(\ln r - \ln h) + q_{m-1}(r,h), \tag{9.2}$$

where

$$\begin{aligned} p_{m-1}(r,h) &= \sum_{2i=0}^{m-1} a_{2i,m} h^{m-1-2i} r^{2i}, \quad |a_{2i,m}| < c_m; \\ q_{m-1}(r,h) &= \sum_{2i=0}^{m-1} b_{2i,m} h^{m-1-2i} r^{2i}, \quad |b_{2i,m}| < c_m. \end{aligned} \tag{9.3}$$

Proof. Relation (9.1) is valid by construction. We shall prove (9.2), (9.3) by induction.

For $m = 1$ they hold true by the definition of ϕ_1. Suppose that they hold true for a certain odd m, i.e.,

$$\phi_m(r,h) = \sum_{2i=0}^{m-1} a_{2i} h^{m-1-2i} r^{2i} (\ln r - \ln h) + b_{2i} h^{m-1-2i} r^{2i}.$$

Let us see what the $(2i)$th terms of the sum become under the transformation

$$\phi_{m+2}(r,h) := \int_h^r \frac{1}{u} \int_h^u t \cdot \phi_m(t,h)\,dt\,du.$$

1. For $r^{2i}(\ln r - \ln h)$ we have (with $c := 1/(2i+2)$)

$$\begin{aligned} s_{2i}(r,h) &:= \int_h^r \frac{1}{u} \int_h^u t^{2i+1}(\ln t - \ln h)\,dt\,du = \int_h^r \frac{1}{u}\left[c u^{2i+2}(\ln u - \ln h) - c^2(u^{2i+2} - h^{2i+2})\right] du \\ &= c^2 r^{2i+2}(\ln r - \ln h) - 2c^3(r^{2i+2} - h^{2i+2}) + c^2 h^{2i+2}(\ln r - \ln h) \\ &= c^2(r^{2i+2} + h^{2i+2})(\ln r - \ln h) - 2c^3(r^{2i+2} - h^{2i+2}), \end{aligned}$$

i.e.,

$$h^{m-1-2i} s_{2i}(r,h) = c^2(h^{m+1-2(i+1)} r^{2(i+1)} + h^{m+1}(\ln r - \ln h) - 2c^3(h^{m+1-2(i+1)} r^{2(i+1)} - h^{m+1}).$$

Consequently,

$$\int_h^r \frac{1}{u} \int_h^u t p_{m-1}(t,h)\,dt\,du = p_{m+1,1}(r,h)(\ln r - \ln h) + q_{m+1,1}(r,h).$$

2. For r^{2i} (with the same $c = 1/(2i+2)$),

$$\int_h^r \frac{1}{u} \int_h^u t^{2i+1} dt\, du = \int_h^r \frac{1}{u} \left(cu^{2i+2} - ch^{2i+2} \right) du = c^2 (r^{2i+2} - h^{2i+2}) - ch^{2i+2} (\ln r - \ln h),$$

whence it follows that

$$\int_h^r \frac{1}{u} \int_h^u t q_{m-1}(t,h)\, dt\, du = p_{m+1,2}(r,h)(\ln r - \ln h) + q_{m+1,2}(r,h).$$

This completes the proof of the lemma.

10. THE CASE (1a) OF THEOREM 2: $l = m$, $p \in (2, \infty]$, $n = 2$

We shall begin with the case of an odd $m = 2m_1 + 1$ from which we shall obtain a construction of an even $m = 2m_1$ as a simple consequence.

10.1. The case $m = 2m_1 + 1$. We set

$$F(r) := \frac{1}{\ln h} (r^2 - H^2)^m \phi_m(r,h), \tag{10.1}$$

so that

$$\frac{\partial^k F(r)}{\partial r^k}\Big|_{r=h,H} = 0, \quad k = 0, 1, \ldots, m-1.$$

Proposition 1a. *The function F admits the decomposition*

$$F(r) = f(r) - \sigma(f, r),$$

where

$$\nabla^{2m} \sigma = \nabla^{2(m_1+1)} \sigma = 0, \quad x \in V; \quad \frac{\partial^k [f - \sigma]}{\partial r^k}\Big|_{r=h,H} = 0, \quad 0 \le k \le m-1. \tag{10.2}$$

In this case,

$$\begin{aligned} f(r) &= c_m (r^2 - H^2)^m r^{m-1} + \frac{1}{\ln h} f_2(r), \\ \sigma(f, r) &= \frac{\ln r}{\ln h} \sum_{2j=0}^{m-1} a_{2j}(H,h) h^{m-1-2j} r^{2j}, \quad |a_0(H,h)| = c'_m(H), \end{aligned} \tag{10.3}$$

and the estimates

$$\begin{aligned} &\|f\|_{W^m_\infty(I^2)} < c_m, \\ &\|\sigma(f)\|_{W^m_p(V)} > \frac{c_m(H)}{h^{1-2/p} |\ln h|} \to \infty, \quad h \to 0,\ p > 2, \end{aligned} \tag{10.4}$$

are valid.

Proof. 1. The boundary conditions in (10.2) are fulfilled by definition. The polyharmonicity of the spline σ, with the order

$$\mu = (m-1)/2 + 1 := m_1 + 1 < m,$$

follows from its representation (10.3).

2. Let us prove (10.3). By virtue of (9.2), (9.3), we have

$$\begin{aligned}\phi_m(r,h) &= p_{m-1}(r,h)(\ln r - \ln h) + q_{m-1}(r,h)\\ &= -p_{m-1}(r,h)\ln h + q_{m-1}(r,h) + p_{m-1}(r,h)\ln r\\ &= c_m r^{m-1}\ln h + \Big[h^2 \ln h p_{m-3}(r,h) + q_{m-1}(r,h)\Big] + p_{m-1}(r,h)\ln r\\ &=: c_m r^{m-1}\ln h + S(r,h) + p_{m-1}(r,h)\ln r\,.\end{aligned}$$

Next we set

$$P(r,H) = (r^2 - H^2)^m.$$

Then we obtain

$$\begin{aligned}F(r) := \frac{1}{\ln h}P(r,H)\phi_m(r) &= \Big[c_m P(r,H) r^{m-1}\Big] + \frac{1}{\ln h}P(r,H)S(r,h) + P(r,H)p_{m-1}(r,h)\frac{\ln r}{\ln h}\\ &=: f_1(r) + \frac{1}{\ln h}f_{21}(r) + Q(r)\frac{\ln r}{\ln h}.\end{aligned}$$

Furthermore,

$$Q(r) := P(r,H)p_{m-1}(r,h) := (r^2 - H^2)^m p_{m-1}(r,h)$$

is an even polynomial of r of degree $2m + (m-1)$, and we decompose it as

$$Q(r) = r^{m+1}T(r,H,h) - q_{m-1}(r,H,h),$$

where

$$\begin{aligned}T(r,H,h) &= \sum_{2j=0}^{m-1} d_{2j}(H,h) r^{2j}, \qquad |d_{2j}(H,h)| < c_m'(H);\\ q_{m-1}(r,H,h) &= \sum_{2j=0}^{m-1} a_{2j}(H,h) h^{m-1-2j} r^{2j}, \qquad |a_0(H,h)| = c_m(H).\end{aligned}$$

Hence,

$$Q(r)\frac{\ln r}{\ln h} = \frac{1}{\ln h}T(r,H,h)r^{m+1}\ln r - q_{m-1}(r,H,h)\frac{\ln r}{\ln h} =: \frac{1}{\ln h}f_{22}(r) - \sigma(f,r).$$

Finally, collecting all the relations and setting $f_2 := f_{21} + f_{22}$, we find that

$$F(r) = f_1(r) + \frac{1}{\ln h}f_2(r) - \sigma(f,r),$$

where

$$\begin{aligned} f_1(r) &= c_m(r^2 - H^2)^m r^{m-1}, \\ f_2(r) &= P(r,H)S(r,h) + T(r,H,h)r^{m+1}\ln r, \\ \sigma(f,r) &= \frac{\ln r}{\ln h}\, q_{m-1}(r,H,h). \end{aligned}$$

We have proved relations (10.3).

3. Let us now prove estimates (10.4). We set

$$\|\cdot\|_{m,p} := \|\cdot\|_{W_p^m(I^2)}.$$

It is obvious that

$$\|f_1\|_{m,\infty} < c_m.$$

Furthermore, since $P(r,H)$, $S(r,h)$, $T(r,H,h)$ are polynomials of r^2 with bounded (by certain c_m) coefficients, and we also have $\|g_m\|_{m,\infty} < c_m$ for $g_m(r) := r^{m+1}\ln r$, we find that

$$\|f_2\|_{m,\infty} \le \|P\|_{m,\infty}\cdot\|S\|_{m,\infty} + \|T\|_{m,\infty}\cdot\|g_m\|_{m,\infty} < c_m.$$

Thus we have

$$\|f\|_{m,\infty} \le \|f_1\|_{m,\infty} + \frac{1}{|\ln h|}\|f_2\|_{m,\infty} < c_m,$$

and the first estimate in (10.4) is proved.

To estimate $\|\sigma(f)\|_{W_p^m(V)}$ from below, we use inequality (6.5)

$$\|g\|^p_{W_p^m(V)} \ge \|D^m g\|^p_{L_p(V)} \ge c^p \int_h^H \Big|\frac{\partial^m g(r)}{\partial r^m}\Big|^p r\,dr.$$

From the representation

$$\sigma(f,r) = \frac{\ln r}{\ln h}\sum_{2j=0}^{m-1} a_{m-1-2j}(H,h)h^{2j}r^{m-1-2j}, \quad |a_0(H,h)| = c_m(H),$$

we find that

$$\frac{\partial^m \sigma(r)}{\partial r^m} = \frac{1}{\ln h}\sum_{2j=0}^{m-1} b_{2j}(H,h)h^{2j}r^{-1-2j}, \quad |b_{m-1}(H,h)| = c_m(H).$$

From this relation, with the substitution $t = r/h$ and for $H/h > 2$, we obtain

$$\int_h^H \Big|\sum_{2j=0}^{m-1} b_{2j}h^{2j}r^{-2j-1}\Big|^p r\,dr = h^{2-p}\int_1^{H/h}\Big|\sum_{2j=0}^{m-1} b_{2j}t^{-2j-1}\Big|^p t\,dt > h^{2-p}|b_{m-1}(H,h)|\epsilon_m,$$

where

$$\epsilon_m := \inf_{c_{2j}\in\mathbb{R}} \int_1^2 |t^{-m} + \sum_{2j=0}^{m-3} c_{2j}t^{-2j-1}|^p t\,dt.$$

The final result is

$$\|\sigma(f)\|_{W_p^m(V)} > \frac{c_m(H)}{h^{1-2/p}|\ln h|} \to \infty, \quad h \to 0,\ p > 2.$$

We have proved Proposition 1a.

10.2. The case $m = 2m_1$. A simple corollary of Proposition 1a is

Proposition 1a′. *For an even $m = 2m_1$ and for any $0 < h < H$ there exists a function $g = g(x)$ such that*

$$\|g\|_{W_\infty^{2m_1}(I^2)} < c_m, \tag{10.5}$$

whereas for the spline $\sigma(g)$ such that

$$\nabla^{2(2m_1)}\sigma = 0, \quad x \in V; \quad \frac{\partial^k[g - \sigma(g)]}{\partial r^k}\Big|_{r=h,H} = 0, \quad 0 \le k \le 2m_1 - 1, \tag{10.6}$$

the limiting process

$$\|D^{2m_1}\sigma(g)\|_{L_p(V)} = \frac{c_m(H)}{h^{1-2/p}|\ln h|} \to \infty, \quad h \to 0,\ p > 2, \tag{10.7}$$

is fulfilled.

Proof. We take the functions f and $\sigma(f)$ from Proposition 1a, i.e., functions such that

$$\nabla^{2(m_1+1)}\sigma(f) = 0, \quad x \in V; \qquad \frac{\partial^k[f - \sigma(f)]}{\partial r^k}\Big|_{r=h,H} = 0, \quad 0 \le k \le 2m_1,$$

and the estimates

$$\|f\|_{W_\infty^{2m_1+1}(I^2)} < c_m,$$

$$\|D^{2m_1+1}\sigma(f)\|_{L_p(V)} > \frac{c_m(H)}{h^{1-2/p}|\ln h|} \to \infty, \quad h \to 0,\ p > 2,$$

are valid. Since $(D^m f)^2 := \sum_{|\alpha|=m} c_\alpha (D^\alpha f)^2$, for every h there exists $\alpha = \alpha_h$, $|\alpha| = 2m_1 + 1$, such that

$$\left\|\frac{\partial^{2m_1+1}\sigma(f)}{\partial x^\alpha}\right\|_{L_p(V)} > \frac{c_m(H)}{h^{1-2/p}|\ln h|} \to \infty, \quad h \to 0,\ p > 2,$$

Let j be the number of any nonzero component $\alpha = (\alpha_1, \ldots, \alpha_n)$. We set

$$g(x) := \frac{\partial f(x)}{\partial x_j}, \quad \sigma(g, x) := \frac{\partial \sigma(f, x)}{\partial x_j}.$$

Then estimates (10.6), (10.7) are obviously valid and the boundary conditions in (10.5) are fulfilled. As to the order of polyharmonicity of $\sigma(g) = \partial\sigma(f)/\partial x_j$, we have, since the differentiation operation preserves this order,

$$\nabla^{2\mu}\sigma(g) = 0, \quad \mu = m_1 + 1 \le 2m_1 = m.$$

We have proved Proposition 1a′ and, together with it, the case (1a) of Theorem 2.

11. THE CASE (1b) OF THEOREM 2: $l = m$, $p \in [1,2)$, $m = 2m_1 + 1$, $n = 2$

For this case, just as for every subsequent new case of Theorem 2, we use some symbols F, f, f_1, f_2, etc., from the previous notation, for the functions that differ from case to case.

We construct the function f that satisfies Theorem 2 under condition (1b) in the form

$$f(r) = \begin{cases} f_1(r), & x \in I_h^2, \\ f_2(r), & x \in U_h, \end{cases} \qquad U_h = \{x:\ 0 \le r \le h\}, \quad I_h^2 = (I^2 \setminus U_h),$$

where

$$\frac{\partial^k f_1(r)}{\partial r^k}\Big|_{r=h} = \frac{\partial^k f_2(r)}{\partial r^k}\Big|_{r=h}, \quad k = 0,\ \dots, m-1,$$

i.e., f will be glued from two pieces. It is obvious that in this case

$$\sigma(f,r) \equiv \sigma(f_1,r),$$

and we shall consider precisely $\sigma(f_1)$. Since the glueing is smooth, the estimate

$$\|f\|_{W_p^m(I^2)} = O_p(1)$$

will follow from (uniform with respect to h) estimates

$$\|f_1\|_{W_p^m(I_h^2)} = O_p(1), \quad \|f_2\|_{W_p^m(I_h^2)} = O_p(1).$$

The necessity of these considerations is connected with our method of constructing f_1 and $\sigma(f_1)$ in which the functions f_1, although bounded in $W_p^m(I_h^2)$ uniformly with respect to $h > 0$, do not belong to $W_p^m(I^2)$ because of the singularities at zero.

In this section we shall construct f_1 and $\sigma(f_1)$.

For $p < 2$, $l = m = 2m_1 + 1$, $n = 2$ we set

$$F(r) := \phi_m(r,H) \cdot \phi_m(r,h), \tag{11.1}$$

so that

$$\frac{\partial^k F(r)}{\partial r^k}\Big|_{r=h,H} = 0, \quad k = 0,1,\ \dots, m-1.$$

Proposition 1b. *The function F admits the decomposition*

$$F(r) = f_1(r) - \sigma(f_1,r),$$

where

$$\nabla^{2m}\sigma = 0, \quad x \in V; \qquad \frac{\partial^k [f_1 - \sigma]}{\partial r^k}\Big|_{r=h,H} = 0, \quad 0 \le k \le m-1. \tag{11.2}$$

In this case,

$$f_1(r) = p_{m-1}(r,H)\,p_{m-1}(r,h)\ln^2 r,$$
$$p_{m-1}(r,t) = \sum_{2j=0}^{m-1} a_{2j}t^{m-1-2j}r^{2j}, \quad |a_{2j}| < c_m,$$
$$\sigma(f_1,r) = O_m(1)r^{2m-2}\ln r\cdot\ln h + \ln r P_{m-2}(r^2) + Q_{m-1}(r^2),$$
$$P_{m-2}\in\pi_{m-2}, \quad Q_{m-1}\in\pi_{m-1}, \tag{11.3}$$

and the estimates

$$\|f_1\|_{W_p^m(I_h^2)} = O_p(1), \quad 1\le p<2,$$
$$\|\sigma(f_1)\|_{L_1(V)} = O_H(|\ln h|)\to\infty, \quad h\to 0, \tag{11.4}$$

are valid.

Proof. 1. The boundary conditions in (11.2) are satisfied by definition (11.1). The polyharmonicity of the spline $\sigma(f_1)$ with order m follows from its representation in (11.3).

2. Let us establish the validity of (11.3). According to Lemma 9.1, we have

$$\phi_m(r,t) = p_{m-1}(r,t)\ln r - p_{m-1}(r,t)\ln t + q_{m-1}(r,t),$$

whence it follows that

$$F(r) := \phi_m(r,h)\phi_m(r,H)$$
$$= [p_{m-1}(r,h)\ln r - p_{m-1}(r,h)\ln h + q_{m-1}(r,h)]$$
$$\times[p_{m-1}(r,H)\ln r - p_{m-1}(r,H)\ln H + q_{m-1}(r,H)]$$
$$=: f_1(r) - \sigma(f_1,r). \tag{11.5}$$

We have set

$$f_1(r) := p_{m-1}(r,h)p_{m-1}(r,H)\ln^2 r,$$

and referred to $\sigma(f_1,r)$ all the other terms which appear after the removal of the parentheses in (11.5).

Since p_{m-1}, q_{m-1} are even polynomials of degree $m-1=2m_1$ of the form

$$p_{m-1}(r,t) = c_{2m_1}r^{2m_1} + \sum_{2j=0}^{2m_1-2} c_{2j}(t)r^{2j}, \quad q_{m-1}(r,t) = d_{m-1}r^{2m_1} + \sum_{2j=0}^{2m_1-2} d_{2j}(t)r^{2j},$$

we find from (11.5) that

$$\sigma(f_1,r) = \sum_{2j=0}^{2m-2}\left(a_{2j}(H,h)r^{2j}\ln r + b_{2j}(H,h)r^{2j}\right), \tag{11.6}$$

the coefficient a_{2m-2} in $r^{2m-2}\ln r$ being equal to

$$a_{2m-2} = -c_{2m_1}^2(\ln H + \ln h) + 2c_{2m_1}d_{2m_1} = O_m(1)\ln h.$$

We have proved relations (11.3).

3. Let us estimate the corresponding norms of f_1 and $\sigma(f_1)$.

For f_1 we have

$$f_1(r) := p_{m-1}(r,H)p_{m-1}(r,h)\ln^2 r,$$

and, since $\|p_{m-1}(\cdot,H)\|_{m,\infty} < c_m$, it suffices to estimate the $W_p^m(I_h^2)$-norm of

$$p_{m-1}(r,h)\ln^2 r = \sum_{2j=0}^{m-1} a_{2j}h^{m-1-2j}r^{2j}\ln^2 r.$$

Let us estimate the $W_p^m(I_h^2)$-norm of every term of the sum with the use of inequality (6.6):

$$\|D^m g\|^p_{L_p(I_h^2)} \le c_m^p \max_{1\le k\le m} \int_h^2 \left|\frac{1}{r^{m-k}}\frac{\partial^k g(r)}{\partial r^k}\right|^p r\,dr.$$

We have (for $p<2$ and $0\le 2j\le m-1$)

$$h^{(m-1-2j)p}\|D^m r^{2j}\ln^2 r\|_p^p \le c_1(m,p)h^{(m-1-2j)p}\int_h^2 r^{(2j-m)p+1}|\ln^{2p} r|\,dr$$

$$\le c_2(m,p)h^{(m-1-2j)p} + c_3(m,p)h^{-p+2}|\ln^5 h| \le c_4(m,p).$$

Thus,

$$\|D^m f_1\|_{L_p(I_h^2)} \le c_{m,p}$$

and since it is obvious that $\|f_1\|_{L_p(I_h^2)} < c'_{m,p}$, we have found estimate (11.4) for f_1.

Let us now estimate the L_1-norm of σ. For this purpose, we introduce the quantity

$$\epsilon_m(H) := \inf_{P_{m-2},Q_{m-1}} \int_{H/2}^{H} \left|r^{2m-2}\ln r + P_{m-2}(r^2)\ln r + Q_{m-1}(r^2)\right| r\,dr.$$

It is clear that

$$\epsilon_m(H) > 0,$$

and actually depends only on m and H. We shall not elucidate the exact order of this quantity.

Since

$$\sigma(f_1,r) = \sum_{2j=0}^{2m-2}\left(a_{2j}r^{2j}\ln r + b_{2j}r^{2j}\right), \quad a_{2m-2} = O_m(1)\ln h,$$

we now have

$$\|\sigma(f_1)\|_{L_1(V)} = c\int_h^H |\sigma(g_1,r)|r\,dr > c\int_{H/2}^H |\sigma(g_1,r)|r\,dr > ca_{2m-2}\epsilon_m(H) > O_m(1)\epsilon_m(H)|\ln h|,$$

i.e.,

$$\|\sigma(f_1)\|_{L_1(V)} = O_{m,H}(|\ln h|) \to \infty, \quad h\to 0.$$

We have proved Proposition 1b.

12. CONTINUATION OF THE CASE (1b) OF THEOREM 2: THE SMOOTHING OF f_1

The functions $f_1(r) := f_1(r; H, h)$ constructed in the preceding section have a bounded W_p^m-norm in I_h^2, i.e., outside of the circle $U_h = \{x : \ r \le h\}$, the bounding being uniform with respect to $h > 0$, but have singularities at zero. In this section we show the possibility of a smooth continuation of f_1 to the interior of the circle U_h.

Lemma 12.1. *For any $h > 0$ there exists a function f_2 such that*

$$\frac{\partial^k f_2(r)}{\partial r^k}\Big|_{r=h} = \frac{\partial^k f_1(r)}{\partial r^k}\Big|_{r=h}, \quad k = 0, 1, \ \dots, m-1;$$
$$\|f_2\|_{W_p^m(U_h)} = O_p(1), \quad 1 \le p < 2.$$

Proof. Since

$$f_1(r) := p_{m-1}(r, H)[p_{m-1}(r, h)\ln^2 r] := f_{11}(r) f_{12}(r),$$

where

$$\|f_{11}\|_{W_\infty^m(I^2)} := \|p_{m-1}(\cdot, H)\|_{W_\infty^m(I^2)} < c_m,$$

it suffices to smooth the second factor

$$f_{12}(r) := p_{m-1}(r, h)\ln^2 r = \ln^2 r \sum_{2j=0}^{m-1} a_{2j} h^{m-1-2j} r^{2j}$$

$$= h^{m-1} \ln^2 r \sum_{2j=0}^{m-1} a_{2j} h^{-2j} r^{2j} =: h^{m-1} J(r). \tag{12.1}$$

Let us construct a polynomial $s(r) = s(r, h)$ such that

$$s(r, h)\Big|_{r=h} = h^{m-1}; \quad \frac{\partial^k s(r, h)}{\partial r^k}\Big|_{r=h} = 0, \quad k = 1, \ \dots, m-1. \tag{12.2}$$

We can define it as

$$s(r, h) := c_m(h) \int_0^r t^{m-2}(t-h)^{m-1} du,$$

where $c_m(h)$ is chosen such that the relation $s(r, h) = h^{m-1}$ is satisfied for $r = h$, and then (12.2) is obviously satisfied. It is easy to show that $s(r, h)$ defined in this way has the form

$$s(r, h) = r^{m-1} \sum_{i=0}^{m-1} b_i h^{-i} r^i.$$

We shall now smooth $f_{12}(r)$ in (12.1) as follows:

$$f_{12}(r) = h^{m-1} J(r) \to s(r, h) J(r) = r^{m-1} \ln^2 r \sum_{i=0}^{2m-2} d_i h^{-i} r^i =: f_{22}(r)\,.$$

According to Leibniz rule, by virtue of (12.2), we immediately obtain

$$\frac{\partial^k f_{22}(r)}{\partial r^k}\Big|_{r=h} = \frac{\partial^k f_{12}(r)}{\partial r^k}\Big|_{r=h}, \quad k = 0, 1, \ldots, m-1.$$

It remains to show that the $W_p^m(U_h)$-norms of the function

$$f_{22}(r,h) = \sum_{i=0}^{2m-2} d_i (r^{m-1}\ln^2 r)(r/h)^i$$

are uniformly bounded with respect to $h > 0$.

For every term of the sum we have

$$|r^{m-1}\ln^2 r| < c_m, \quad |(r/h)^i| < 1, \quad r < h,$$

i.e., $f_{22} \in L_\infty(U)$.

In order to estimate the L_p-norms of the mth derivatives, we use Leibniz rule and the relations

$$\begin{aligned} \|D^{m-k}(r/h)^i\|_{L_\infty(U)} &\le c_m h^{-(m-k)}, \\ \|D^k r^{m-1}\ln^2 r\|_{L_p(U)} &\le c_m h^{m-1-k+2/p}\ln^2 h, \end{aligned}$$

and this yields

$$\|D^m f_{22}\|_{L_p(U_h)} \le c_m h^{-1+2/p}\ln^2 h \to 0, \quad h \to 0,\ p < 2.$$

We have proved Lemma 12.1.

We have thus constructed the function $f \in W_p^m(I^2)$ that satisfies all requirements (7.2)–(7.4) of Theorem 2 in the case (1b), but it is necessary, in addition, that the function f should also be from $W_2^m(I^2)$. However, it is obvious that we can now smooth the function f in a small neighborhood U_ϵ of zero to $\tilde f \in W_2^m(I^2)$ so that we should have

$$\text{(i)} \quad \tilde f(x) = f(x),\ x \in I^2 \setminus U_\epsilon; \qquad \text{(ii)} \quad \|\tilde f - f\|_{W_p^m(I^2)}\| < \epsilon$$

for any $\epsilon > 0$. Then, for $\epsilon < h$ the function $\tilde f$ will satisfy all requirements of Theorem 2 for the case (1b).

Indeed, condition (i) will imply the relation $\sigma(\tilde f) = \sigma(f)$ and, hence, all properties of the spline will remain unchanged, and, by virtue of (ii), the norm of $\tilde f$ will also be bounded.

We have proved Theorem 2 for the case (1b).

13. THE CASE (1d) OF THEOREM 2: $l = m$, $p \in [1,2)$, $m = 2m_1$, $n = 4$

In this case the construction of the function f is a slight modification of the case (1b). We shall show that for the function

$$f_1(r) := r^{2m_1-2}\ln^2 r + \ldots, \qquad f \in W_{2-\epsilon}^{2m_1}(I^4) \quad \forall \epsilon > 0,$$

the Hermite spline $\sigma(f_1)$ on the ring V has the form

$$\sigma(f_1, r) = r^{2m_1-2} \ln r \cdot \ln h + \dots,$$

whence follows the unboundedness of $\|\sigma(f_1)\|_{L_1(V)}$ as $h \to 0$.

The smoothing of f_1 in the neighborhood of zero can be carried out in the same way as in Sec. 11.

13.1. Auxiliary lemmas. We set

$$\psi_2(r, H) := 2H^2(\ln r - \ln H) - (r^2 - H^2),$$

so that

$$\frac{\partial^k \psi_2(r)}{\partial r^k}\Big|_{r=H} = 0, \quad k = 0, 1,$$

and, for even $m = 2m_1 = 4, 6, \dots,$ define

$$\psi_m(r, H) := \int_H^r \frac{1}{u} \int_H^u t \cdot \psi_{m-2}(t, H)\, dt\, du.$$

Lemma 13.1. *The function ψ_m satisfies the relations*

$$\frac{\partial^k \psi_m(r, H)}{\partial r^k}\Big|_{r=H} = 0, \quad k = 0, \dots, m-1, \tag{13.1}$$

and has the form

$$\psi_m(r, H) = H^2 p_{m-2}(r, H)(\ln r - \ln H) + q_m(r, H), \tag{13.2}$$

where

$$\begin{aligned} p_{m-2}(r, H) &= \sum_{2i=0}^{m-2} a_{2i,m} H^{m-2i} r^{2i}, \quad |a_{2i,m}| \sim c_m; \\ q_m(r, H) &= \sum_{2i=0}^{m} b_{2i,m} H^{m-2-2i} r^{2i}, \quad |b_{2i,m}| \sim c_m. \end{aligned} \tag{13.3}$$

The proof is similar to that of Lemma 9.1 for ϕ_m.

Next, for the function $\phi_l(r, h)$ defined in (9.1)–(9.3), for odd $l = 2l_1 + 1$ we set

$$\chi_m(r, h) := \chi_{2m_1}(r, h) := \frac{1}{r} \frac{\partial \phi_{2m_1+1}(r, h)}{\partial r}.$$

Lemma 13.2. *The function χ_m satisfies the relations*

$$\frac{\partial^k \chi_m(r, h)}{\partial r^k}\Big|_{r=h} = 0, \quad k = 0, \dots, m-1, \tag{13.4}$$

and has the form

$$\chi_m(r, h) = s_{m-2}(r, h)(\ln r - \ln h) + t_{m-2}(r, h) + b_m h^m r^{-2}, \tag{13.5}$$

where

$$\begin{aligned} s_{m-2}(r,h) &= \sum_{2i=0}^{m-2} c_{2i,m} h^{m-2-2i} r^{2i}, \quad |c_{2i,m}| \sim c_m; \\ t_{m-2}(r,h) &= \sum_{2i=0}^{m-2} d_{2i,m} h^{m-2-2i} r^{2i}, \quad |d_{2i,m}| \sim c_m. \end{aligned} \tag{13.6}$$

Proof. This lemma is a direct corollary of Lemma 9.1 for ϕ_{2m_1+1}.

13.2. Construction of f_1. We set

$$F(r) := \psi_m(r,H)\chi_m(r,h),$$

so that

$$\frac{\partial^k F(r)}{\partial r^k}\Big|_{r=H,h} = 0, \quad k = 0, \ldots, m-1. \tag{13.7}$$

Proposition 1d. *The function F admits the decomposition*

$$F(r) = f_1(r) - \sigma(f_1, r),$$

where

$$\nabla^{2m}\sigma = 0, \quad x \in V; \qquad \frac{\partial^k [f_1 - \sigma]}{\partial r^k}\Big|_{r=h,H} = 0, \quad 0 \le k \le m-1. \tag{13.8}$$

In this case,

$$\begin{aligned} f_1(r) &= H^2 p_{m-2}(r,H)\, s_{m-2}(r,h) \ln^2 r + c_m H^m h^m r^{-2} \ln r + c'_m r^{2m-2} \ln r, \\ \sigma(f_1, r) &= O_m(1) r^{2m-4} \ln r \ln h + c(H,h) r^{-2} + P_{m-3}(r^2) \ln r + Q_{m-1}(r^2), \end{aligned} \tag{13.9}$$

and the estimates

$$\begin{aligned} \|f_1\|_{W_p^m(I_h^4)} &= O_p(1), \quad p \in [1,2); \\ \|\sigma(f_1)\|_{L_1(V_h)} &= O_H(|\ln h|) \to \infty, \quad h \to 0, \end{aligned} \tag{13.10}$$

are valid.

The proof is similar to that of Proposition 1b from Sec. 11. Using relations (13.2), (13.5) for ψ_m and χ_m respectively, we write out the product in the relation

$$F(r) := f_1(r) - \sigma(f_1, r) := \psi_m(r)\chi_m(r)$$

and refer to $\sigma(f_1)$ all m-harmonic terms, i.e., the terms that enter (for $n = 4$) into the collection

$$r^{-2}, \ \{r^{2j} \ln r\}_{j=0}^{m-2}, \ \{r^{2j}\}_{j=0}^{m-1}.$$

All the other terms constitute f_1. In this way we obtain (13.9).

Conditions (13.8) are fulfilled automatically.

Estimate (13.10) can be easily derived from (13.9). The estimate for $\|\sigma(f_1)\|_{L_1(V)}$ is obvious because of the factor $\ln h$. We can estimate the norm of f_1 as follows.

For the factors of the first term of f_1 in (13.9) we have

$$\|p_{m-2}(\cdot, H)\|_{W^m_\infty(I^4)} < c_m, \quad \|s_{m-2}(\cdot, h)\ln^2(\cdot)\|_{W^m_p(I^4_h)} < c_m.$$

The first inequality is obvious; the second inequality is valid since

$$s_{m-2}(r, h)\ln^2 r = \sum_{2j=0}^{m-2} c_{2j} h^{m-2-2j} r^{2j} \ln r$$

and the $W^m_p(I^4_h)$-norm of each term is bounded. This boundedness can be established in the same way as for the second term of f_1 in (13.9) for which we have (for $p \in [1, 2)$ and $n = 4$)

$$h^{mp}\|D^m r^{-2}\ln r\|_{L_p(I^4_h)} \le c_1(m,p) h^{mp} \int_h^2 r^{(-2-m)p+3} \ln^p r\, dr$$
$$\le c_2(m,p)h^{mp} + c_3(m,p)h^{-2p+4}|\ln^3 h| \le c_4(m,p)\,.$$

Finally, the third term of f_1 in (13.9), namely, $r^{2m-2}\ln r$, is obvious from $W^m_\infty(I^4)$.

We have proved Proposition 1d.

13.3. Smoothing of f_1. This smoothing can be carried out in the same way as in the case (1b), (see Sec. 12).

We have proved the case (1d) of Theorem 2.

14. THE CASE (2a) OF THEOREM 2: $l = m+1$, $p = 1$, $m = 2m_1$, $n = 4$

This case follows immediately from the results of Sec. 13, namely, it is easy to verify that the relation

$$\|f_1\|_{W_1^{m+1}(I^4_h)} = O(1)$$

is satisfied uniformly with respect to $h > 0$ for the functions $f_1 := f_{1;H,h}$ defined in (13.9). For instance, for the same term $c_m H^m h^m r^{-2}\ln r$ in representation (13.9) for $f_1(r)$ we have (for $p = 1$ and $n = 4$)

$$h^m\|D^{m+1} r^{-2}\ln r\|_{L_1(I^4_h)} \le c_1(m) h^m \int_h^2 r^{-2-(m+1)+3}|\ln r|\, dr$$
$$\le c_2(m)h^m + c_3(m)h|\ln h| \le c_4(m)h|\ln h| \to 0, \quad h \to 0.$$

The smoothing of $f_1 \in W_1^{m+1}(I^4_h)$ to the function $f \in W_1^{m+1}(I^4) \cap W_2^m(I^4)$ with the preservation of the order of the W_1^{m+1}-norm is carried out as before.

In this way we can prove the case (2a) of Theorem 2.

15. THE CASE (1c) OF THEOREM 2: $l = m$, $p \in [1, 3/2)$, $n = 3$

In the case of an odd dimension n, we cannot use the results which we obtained earlier for an even n since the nature of polyharmonic functions is different. However, this case is very simple.

We set

$$F(r) := \frac{1}{r}(r-H)^m(r-h)^{m-1}(\ln r - \ln h). \tag{15.1}$$

Proposition 1c. *The function F admits the decomposition*

$$F(r) = f_1(r) - \sigma(f_1, r),$$

where

$$\nabla^{2m}\sigma = 0, \quad x \in V; \qquad \frac{\partial^k[f_1 - \sigma]}{\partial r^k}\Big|_{r=h,H} = 0, \quad 0 \le k \le m-1. \tag{15.2}$$

In this case,

$$\begin{aligned} f_1(r) &= r^{-1}(r-H)^m(r-h)^{m-1}\ln r; \\ \sigma(f_1, r) &= r^{-1}(r-H)^m(r-h)^{m-1}\ln h, \end{aligned} \tag{15.3}$$

and the estimates

$$\begin{aligned} \|f_1\|_{W_p^m(I_h^3)} &= O_p(1), \quad p \in [1, 3/2); \\ \|\sigma(f_1)\|_{L_1(V)} &= O_H(|\ln h|) \to \infty, \quad h \to 0, \end{aligned} \tag{15.4}$$

are valid.

Proof. All relations are obvious, except for the estimates $\|f_1\|_{W_p^m(I_h^3)}$ in (15.4). Let us prove their validity.

Since the factor $(r-H)^m$ in representation (15.3) for f_1 is bounded in $W_\infty^m(I^3)$, it suffices to prove the boundedness in the norm $W_p^m(I_h^3)$ of the factor

$$r^{-1}(r-h)^{m-1}\ln r = \sum_{j=0}^{m-1} a_j h^{m-1-j} r^{j-1}, \quad |a_j| < c_m.$$

For the terms in the sum we have (for $p \in [1, 3/2)$, $0 \le j \le m-1$, and $n = 3$)

$$\begin{aligned} h^{(m-1-j)p}\|D^m r^{j-1}\ln r\|^p_{L_p(I_h^3)} &\le c_1(m,p)h^{(m-1-j)p}\int_h^2 r^{(j-1-m)p+2}\ln^p r\, dr \\ &\le c_2(m,p)h^{(m-1-j)p} + c_3(m,p)h^{-2p+3}|\ln h|^{5/2} \le c_4(m,p). \end{aligned}$$

We have proved Proposition 1c.

It remains to smooth f_1 to $f \in W_p^m(I^3) \cap_2^m (I^3)$, and then the case (1c) of Theorem 2 will be proved.

16. THE CASE (2b) OF THEOREM 2: $l = m+1$, $p = 1$, $n = 5$

This case is also simple. We set

$$G(r) := (r-H)^m (r-h)^{m-1}(r^2 + Ar + B).$$

We choose the coefficients A, B such that the polymonial $G(r)$ would not include the monomials r and r^{2m}. Let

$$G(r) = \sum_{i=0}^{2m+1} a_i r^i.$$

Then we have

$$a_{2m} = A - mH - (m-1)h = 0,$$
$$a_1 = H^{m-1}h^{m-2}(A \cdot Hh - B(mh + (m-1)H)) = 0,$$

whence it follows that

$$A = O_m(H), \quad B = O(h).$$

Consequently,

$$G(r) = c_{2m+1}r^{2m+1} + \sum_{j=m+1}^{2m-1} c_j(H,h)r^j + \sum_{j=2}^{m} c_j(H,h)h^{m-j}r^j + c_1(H,h)h^m, \quad |c_j(H,h)| < c'_m. \tag{16.1}$$

Now we set

$$F(r) := \frac{1}{r^3}G(r)(\ln r - \ln h),$$

so that

$$\frac{\partial^k F(r)}{\partial r^k}\Big|_{r=H,h} = 0, \quad k = 0, 1, \ldots, m-1.$$

Proposition 2b. *The function F admits the decomposition*

$$F(r) = f_1(r) - \sigma(f_1, r),$$

where

$$\nabla^{2m}\sigma = 0, \quad x \in V; \qquad \frac{\partial^k [f_1 - \sigma]}{\partial r^k}\Big|_{r=h,H} = 0, \quad 0 \le k \le m-1. \tag{16.2}$$

In this case,

$$f_1(r) = r^{-3}G(r)\ln r,$$
$$\sigma(f_1, r) = r^{-3}G(r)\ln h \tag{16.3}$$

and the estimates

$$\|f_1\|_{W_1^{m+1}(I_h^5)} = O(1),$$
$$\|\sigma(f_1)\|_{L_1(V)} = O_H(\ln h) \to \infty, \quad h \to 0, \tag{16.4}$$

are valid.

Proof. Representation (16.3) is obvious.

From relation (16.1) we find that $r^{-3}G(r)$ is a linear combination of the functions

$$\{r^{2j-5},\, r^{2j-2}\}_{j=1}^{m}$$

which are polyharmonic with order $j \le m$, i.e., the function $\sigma(f_1)$ is m-harmonic. The boundary conditions are fulfilled by the definition of F, and therefore relations (16.2) hold true.

Let us prove estimates (16.4). For the $L_1(V)$-norm of $\sigma(f_1)$ the estimate is obvious because of the factor $\ln h$. Let us find the estimate for the norm of f_1. From (16.1), (16.3) we find that

$$f_1(r) = P_m(r, H, h) r^{m-2} \ln r + \sum_{j=0}^{m} d_j(H, h) h^{m-j} r^{j-3} \ln r.$$

The first term in this representation is obviously from $W_1^{m+1}(I^5)$. For the terms of the sum we have (for $p = 1$, $0 \le j \le m$, and $n = 3$)

$$h^{m-j} \|D^{m+1} r^{j-3} \ln r\|_{L_1(I_h^5)} \le c_1(m) h^{m-j} \int_h^2 r^{j-3-(m+1)+4} |\ln r|\, dr$$
$$\le c_2(m,p) h^{m-j} + c_3(m) h |\ln h| \le c_4(m,p).$$

We have proved Proposition 2b.

We again smooth f_1 to $f \in W_1^{m+1}(I^5) \cap W_2^m(I^5)$, and this completes the proof of the case (2b) of Theorem 2 and, together with it, the whole Theorem 2.

17. THE CASE (3) OF THEOREM 1′: $l = m - 1$, $p = \infty$, $n = 4$

We have not managed to reduce this case to the preceding scheme for $l = m, m+1$ in which we used Hermite's D^m-splines that interpolate the values of the function f and those of its partial derivatives on manifolds of dimension $n - 1$. However, this case reduces to a multiple Hermite interpolation at the points (on manifolds of dimension 0).

17.1. Reduction to multiple D^m-splines. According to Sobolev's embedding theorem, we have

$$m - n/2 > k \ge 0 \quad \Rightarrow \quad W_2^m(I^n) \to C^k(I^n)$$

and, hence, in the definition of the D^m-spline as the solution of the variational problem

$$\|D^m g\|_{L_2(I^n)} \to \min$$

under interpolation constraints, we can define as these interpolation constraints not only the values of $g(x)$ at the points $t_i \in \Delta$ but also the values of the partial derivatives $D^\alpha g(x)$ up to the order k inclusive.

To be more precise, for $0 \le k < m - n/2$ for the multi-indices $\alpha \in \mathbb{Z}_+^n$ we set

$$\mathcal{A}_k := \{\alpha : \ |\alpha| \le k\}$$

and understand the net Δ_A as the collection of pairs

$$\Delta_A := (\Delta, A) := \{(t_i, A_i)\}_{i=1}^N,$$

where

$$\Delta = \{t_i\}, \quad t_i \in I^n, \qquad A = \{A_i\}, \quad A_i \subset \mathcal{A}_k.$$

Then, for any $f \in W_2^m(I^n)$, we can define the *multiple* D^m-spline $s_A(f)$ as the solution of the problem

$$\begin{aligned} s_A(f) &= s(f, m, \Delta_A, \Omega) \\ &= \arg\ \min\{\, \|D^m g\|_2 :\ g \in W_2^m(\Omega),\ D^\alpha g(t_i) = D^\alpha f(t_i),\ \alpha \in A_i,\ 1 \le i \le N\}\,. \end{aligned}$$

The existence and uniqueness of $s_A(f)$ are known from the general theory of variational splines. For the ordinary spline $s(f)$, we have

$$s(f) = s_A(f), \quad A = \mathcal{A}_0.$$

The following lemma allows us to reduce the original problem on the unconditional convergence of discrete D^m-splines in $C^{m-1}(\Omega)$ to a similar problem on multiple D^m-splines. We give it without proof, and only for the case $n = 4$.

Lemma 17.1. *Let $n = 4$, $m > 2\,(= n/2)$, $k_* = m - 3\,(< m - n/2)$. Suppose, furthermore, that $e_1 = (1,0,0,0)$ is a unit vector in $\mathbb{R}^4$, and for the arbitrary $t_i \in I^4$, $\eta > 0$ we have*

$$\delta_\eta = \{t_1 + j\eta e_1\}_{j=0}^{k_*}, \quad \delta_2 = \{t_i\}_{i=2}^N, \quad \Delta_\eta = \delta_\eta \cup \delta_2;$$
$$\Delta = \{t_1\} \cup \Delta_2, \quad A_1 = \{je_1\}_{j=0}^{k_*}, \quad A_i = \{\mathbf{0}\}.$$

Finally, suppose that

$$s_\eta := s(f, m, \Delta_\eta, I^4),$$
$$s_A := s(f, m, \Delta_A, I^4) :=$$
$$:= \arg\ \min\left\{\, \|D^m g\|_2 :\ g \in W_2^m(I^4),\ \frac{\partial^j g}{\partial x_1^j} = \frac{\partial^j f}{\partial x_1^j}|_{x=t_1},\ 0 \le j \le k_*\ g|_{\Delta_2} = f|_{\Delta_2} \right\}$$

for an arbitrary $f \in W_2^m(I^4)$.

Then

$$\|s_A - s_\eta\|_{W_2^m(I^4)} \to 0, \quad \eta \to 0.$$

17.2. Unboundedness of multiple D^m-splines in C^{m-1}. It follows from Lemma 17.1 and Lemma 5.2 that on any ball $B(a, \epsilon)$ such that

$$B(a, 2\epsilon) \subset V := I^4 \setminus \{\Delta_\eta\}_{\eta < \eta_0},$$

we have the convergence

$$\|s_A - s_\eta\|_{C^{m-1}[B(a,\epsilon)]} \to 0, \quad \eta \to 0.$$

Thus, in order to prove the last case 3) of Theorem $1'$, it suffices to prove the unboundedness of s_A in $C^{m-1}(I^n)$.

We shall prove the following statement.

Theorem 3. *Let the condition*

$$(3) \qquad l = m-1, \quad p = \infty, \quad n = 4$$

be fulfilled. Suppose, furthermore, that $k_* = m-3$ and the arbitrary net

$$\Delta_A = \{t_i, A_i\}_1^N, \quad t_i \in \Delta, \quad A_i \in A, \quad \max_i \max_{\alpha \in A_i} |\alpha| = k_*,$$

is defined.

Then, for any $f \in C^{m-1} \cap W_2^m(I^4)$ and any $M > 0$ there exists a ball $B(a, 2\epsilon)$ such that

$$B(a, 2\epsilon) \subset I^4 \setminus \Delta, \quad \|D^{m-1} s_A(f)\|_{L_\infty[B(a,\epsilon)]} > M. \tag{17.1}$$

Proof. For $s_A(f)$ we have a representation similar to (2.1) for $s(f)$, namely, for $A = \{A_i\}$ with $A_i \in \mathcal{A}_k$, $k < m - n/2$, we have

$$s_A(f, x) = \sum_{i=1}^N \sum_{\alpha \in A_i} c_{i\alpha} D^\alpha G(x - t_i) + F(x), \tag{17.2}$$

where, as before, we have

$$G(x) = \begin{cases} |x|^{2m-n}, & n = 2n_1; \\ |x|^{2m-n} \ln |x|, & n = 2n_1 + 1, \end{cases} \tag{17.3}$$

and the function $F(x) = F(x; m, \Delta_A, \Omega)$ is polyharmonic in I^n.

It follows from (17.3) that for $|\alpha| = k$ (and for the agreement that $W_\infty^r = C^r$)

$$D^\alpha G(x) \in W_p^l(\mathbb{R}^n, \mathrm{loc}) \quad \Leftrightarrow \quad l - n/p < 2m - n - k, \tag{17.4}$$

and

$$\operatorname{ess\,sup} |D^\alpha G(x)| = \begin{cases} \infty, & n = 2n_1, \\ O(1), & n = 2n_1 + 1 \end{cases} \tag{17.5}$$

for $p = \infty$, $l = 2m - n - k$ in the neighborhood of zero $|x| < \epsilon$.

Let us find the worse smoothness of $D^\alpha G$. We can see from (17.4) that the smoothness of G decreases with the growth of k, but in the definition of multiple splines we are restricted by the inequality

$$k < m - n/2.$$

Under this restriction, the maximal value of $k = k_*$ is

$$k_* = m - (n/2 + 1), \quad n = 2n_1;$$

$$k_* = m - (n+1)/2, \quad n = 2n_1 - 1,$$

whence, for any n, we have

$$2m - n - k_* = m - (n_1 - 1) = n - \left[\frac{n-1}{2}\right]. \tag{17.6}$$

Thus, we find from (17.2)–(17.6) that

$$s_A(f) \in W_p^l(I^n, \mathrm{loc}) \quad \Leftrightarrow \quad l - n/p < m - \left[\frac{n-1}{2}\right] \tag{17.7}$$

under the interpolation of the partial derivatives of $f \in W_2^m(I^n)$ of the maximum possible order $k = k_*$ at some points of the net Δ_A.

In the case $n = 4$, $l = m - 1$ which is of interest to us we have, for $k_* = m - 3$,

$$s_A(f) \in W_p^{m-1}(I^4, \mathrm{loc}) \quad \Leftrightarrow \quad m - 1 - n/p < m - 1.$$

With due account of (17.5), we infer that if, for $p = \infty$, the set A_i in the pair $(t_i, A_i) \in \Delta_A$ contains α with $|\alpha| = k_* = m - 3$, then

$$\|D^{m-1} s_A(f)\|_{L_\infty[B(t_i,\rho)]} = \infty \quad \forall \rho > 0.$$

Now, for any $M, \epsilon > 0$, there exists a point $a \in B(t_i, 3\epsilon)$ such that

$$|D^\beta s_A(f, a)| > M$$

for a certain multi-index β, $|\beta| = m - 1$, and this gives the estimate of the norm in (17.1).

Now if $3\epsilon < \underline{h}(\Delta) := \inf |t_i - t_j|$, then $B(a, 2\epsilon) \cap \Delta = \varnothing$, i.e., the first requirement imposed on B in (17.1) is also met. We have proved Theorem 3.

18. COMMENTS

18.1. Possibility of a complete solution. As was pointed out in Sec. 4, in order to prove Conjecture 1, namely, the fact that

$$s_\nu(f, m) \to f \quad \text{uncond. in } W_p^l(I^n) \quad \Leftrightarrow \quad (l, p) = (m, 2)$$

for $n \geq 2$, it suffices to construct examples of divergence of discrete D^m-splines only in the following three cases:

$$\begin{array}{llll} \text{(i)} & l = m, & p \in [1, 2), \quad m = 2m_1, & n = 2; \\ \text{(ii)} & l = m + 1, & p = 1, & n = 2; \\ \text{(iii)} & l = m - 1, & p = \infty, & n = 2. \end{array} \tag{18.1}$$

We have shown that this problem reduces to a similar problem for Hermite D^m-splines, which are nothing other than the solution of the general Dirichlet problem for a polyharmonic operator in a certain domain Ω with the boundary Δ:

$$\nabla^{2m} \sigma = 0, \quad x \in \Omega \setminus \Delta, \qquad \sigma\Big|_\Delta = f\Big|_\Delta.$$

Let

$$T = T_{l,p}(\Omega): \ W_p^l(\Omega) \to W_p^l(\Omega), \quad T(f) = \sigma(f)$$

be an operator of polyharmonic continuation of the boundary values of $f \in W_p^l(\Omega)$ to the whole domain. Then, in order to prove Conjecture 1, it suffices to show that

$$\sup_{\Omega} \|T_{l,p}(\Omega)\| = \infty, \quad (l,p) \neq (m,2), \quad n \geq 2.$$

Except for certain pairs (l,p), we have proved this fact by considering the simplest domains:

(1) a ring for $l = m, m+1$,

(2) actually a ball with the deleted center for $l = m-1$.

It is very likely that slightly more exotic domains (e.g., a circle with a deleted segment) will give counterexamples for the required cases of (18.1).

18.2. Generalization of the problem. We can pose a more general question concerning the necessary and sufficient conditions under which

$$s_\nu(f) \to f \quad \text{uncond. in } W_q^k(\Omega), \quad f \in W_p^l(\Omega). \tag{18.2}$$

According to the results obtained, we can expect that (18.2) holds if and only if the embeddings

$$W_p^l(\Omega) \to W_2^m(\Omega) \to W_q^k(\Omega) \tag{18.3}$$

are simultaneosly satisfied.

Our results allow us also to suppose that

(1) if the first embedding in (18.3) is not satisfied, then there can be found an example in which $s_\nu(f)$ diverge already in $L_1(\Omega)$;

(2) now if the first embedding exists and the second does not, then there can be found an example in which $s_\nu(f)$ diverge not only in $W_q^k(\Omega)$ but also in any other space $W_{q'}^{k'}(\Omega)$ that does not include $W_2^m(\Omega)$.

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REFERENCES

1. De Boor, C., The Quasi-Interpolant as a Tool in Elementary Polynomial Spline Theory, *Approx. Theory*, Lorentz, G.G., Ed., New York: Academic Press, 1973, pp. 269–276.
2. De Boor, C., On Bounding Spline Interpolation, *Approx. Theory*, 1975, vol. 14, pp. 191–203.
3. De Boor, C., On a Max-Norm for the Least-Squares Spline Approximant, in *Approximation and Function Spaces*, Ciesielsky, Z., Ed., New York, 1981, pp. 163–175.
4. Zmatrakov, N.L., Convergence of the Third Derivatives of Interpolation Cubic Slines in the Metrics L_p $(1 \leq p \leq \infty)$, *Mat. Zametki*, 1981, vol. 30, no. 1, pp. 83–99.
5. Zmatrakov, N.L. and Subbotin, Yu.N., Multiple Interpolation Splines of Degree $2k+1$ with Defect k, *Trudy Mat. Inst. Akad. Nauk SSSR*, 1983, vol. 164, pp. 75–99.

6. *Levin, A. and Dyn, N.,* Construction of Surface Spline Interpolants of Scattered Data over Finite Domain, *RAIRO Anal. Numer.*, 1982, vol. 16, no. 3. pp. 201–209.

7. Matveev, O.V., Spline-Interpolation of Functions in Several Variables and Bases in Sobolev Spaces, *Trudy Mat. Inst. Ross. Akad. Nauk*, 1992, vol. 198, pp. 125–152.

8. Matveev, O.V., Approximation Properties of Interpolation D^m-Splines, *Dokl. Akad. Nauk SSSR*, 1991, vol. 321, no. 1, pp. 14–18.

9. Nord, S., Approximation Properties of the Spline Fit, *BIT*, 1967, vol. 7, pp. 132–144.

10. Shadrin, A.Yu., On L_p-Boundedness of the L_2-Projector onto Splines, *Approx. Theory*, 1994, vol. 77, pp. 331–348.

11. *Ciesielsky, Z.,* Properties of the Orthomormal Franklin System, II, *Studia Math.*, 1966, vol. 27, pp. 289–323.

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