On k-monotone approximation by free knot splines

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Abstract

Let $\mathcal{S}_{N,r}$ be the (nonlinear) space of free knot splines of degree $r-1$ with at most N pieces in $[a,b]$, and let $\mathcal{M}^k$ be the class of all k-monotone functions on (a,b), *i.e.,* those functions f for which the kth divided difference $[x_0,\dots,x_k]f$ is nonnegative for all choices of $(k+1)$ distinct points $x_0,\dots,x_k$ in (a,b).

In this paper, we solve the problem of *shape preserving* approximation of k-monotone functions by splines from $\mathcal{S}_{N,r}$ in the $\mathbb{L}_p$-metric, *i.e.,* by splines which are constrained to be k-monotone as well. Namely, we prove that the order of such approximation is essentially the same as that by the non-constrained splines. Precisely, it is shown that, for every $k,r,N\in\mathbb{N}$, $r\ge k$, and any $0<p\le\infty$, there exist constants $c_0=c_0(r,k)$ and $c_1=c_1(r,k,p)$ such that

$$\operatorname{dist}(f,\mathcal{S}_{c_0N,r}\cap\mathcal{M}^k)_p \le c_1 \operatorname{dist}(f,\mathcal{S}_{N,r})_p \quad \forall f\in\mathcal{M}^k .$$

This extends to all $k\in\mathbb{N}$ results obtained earlier by Leviatan & Shadrin and by Petrov for $k\le 3$.

1 Introduction and Main Results

In this paper, we solve the problem of *shape preserving* approximation of k-monotone functions by splines with free knots in the $\mathbb{L}_p$-metric, *i.e.,* by splines which are constrained to be k-monotone as well. Namely, we prove that the order of such approximation is essentially the same as that by the non-constrained splines, confirming thus expectations of some standing.

Given $k\in\mathbb{Z}_+$ and an interval $I=(a,b)$, a function $f: I\mapsto\mathbb{R}$ is said to be k-monotone on I if its kth divided differences $[x_0,\dots,x_k]f$ are nonnegative for all choices of $(k+1)$ distinct points $x_0,\dots,x_k$ in I. We denote the class of all such functions by $\mathcal{M}^k:=\mathcal{M}^k(I)$. Thus, $f\in\mathcal{M}^0$ is non-negative, $f\in\mathcal{M}^1$ is non-decreasing, and $f\in\mathcal{M}^2$ is a convex function. If $f\in\mathbb{C}^k(I)$, then $f\in\mathcal{M}^k$ if and only if $f^{(k)}\ge 0$ on I.

We would like to emphasize that functions from $\mathcal{M}^k$ are not assumed to be defined at the endpoints of the interval (a,b), and, hence, have to be neither bounded nor integrable on (a,b). For example, if $f(x)=(-1)^k x^{-1-1/p}$, then $f\in\mathcal{M}^k(0,1)$ for $k\in\mathbb{N}$, but $f\notin\mathbb{L}_p(0,1)$, $0<p\le\infty$. (Throughout the paper, $\mathbb{L}_\infty(I)$ denotes the space of all measurable essentially bounded functions equipped with the norm $\|f\|_{\mathbb{L}_\infty(I)}:=\operatorname{ess\,sup}_I|f|$.)

Hence, we now define $\mathcal{M}^k_p:=\mathcal{M}^k\cap\mathbb{L}_p$, and also remark that the functions from the cone $\mathcal{M}^k$ are sometimes referred to as "k-convex".

Let $f\in\mathcal{M}^k_p$ and $\mathcal{U}$ be a subset of $\mathbb{L}_p$. The best (non-constrained) approximation of f from $\mathcal{U}$ is defined by

$$E(f,\mathcal{U})_p := \inf_{u\in\mathcal{U}}\|f-u\|_p .$$

In contrast, in k-monotone approximation , one is interested in the value

$$E^{(k)}(f,\mathcal{U})_p := \inf_{u\in\mathcal{U}\cap\mathcal{M}^k}\|f-u\|_p.$$

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That is, approximants are assumed to preserve the k-monotone shape of f. Clearly, the shape preserving approximation is more restrictive, hence, $E^{(k)}(f,\mathcal{U})_p \ge E(f,\mathcal{U})_p$ for all $f \in \mathcal{M}^k$ and $\mathcal{U} \subset \mathbb{L}_p$. Is it much worse? Lorentz & Zeller [10] proved that, for $\mathcal{U} = \Pi_n$, the space of all algebraic polynomials of order n, any $k \in \mathbb{N}$, and any constant $c_0 \in \mathbb{N}$, there exists a function $f \in \mathcal{M}_p^k$ such that

$$\frac{E^{(k)}(f,\Pi_{c_0 n})_p}{E(f,\Pi_n)_p} \to \infty \quad \text{as} \quad n \to \infty\,. \tag{1.1}$$

(The same estimate is true for a sequence of any reasonable linear subspaces $\mathcal{U}_n$ instead of Π_n.) On the other hand, monotone and convex polynomial approximations allow Jackson type estimates, for example,

$$E^{(k)}(f,\Pi_n)_\infty \le c_k \omega_{k+1}(f,\tfrac{1}{n})_\infty\,, \quad k=1,2,$$

but they have essential restrictions (as well as gaps) in comparison with the non-constrained estimates.

Splines with free knots, $s \in \mathcal{S}_{N,r}$, are piecewise polynomials of order r (degree $r-1$) where only the number of pieces, N at most, not their position, is being prescribed. (Note that we do not make any assumptions about the smoothness of functions in $\mathcal{S}_{N,r}$.) They are a classical tool of *non-linear* approximation (along with the rational functions). As that, they achieve a better rate of approximation compared with the linear methods. The simplest example (see *e.g.* [4, p. 365]) is that

$$E(f,\mathcal{S}_{N,1})_\infty \le \frac{K}{2N} \quad \Leftrightarrow \quad \mathrm{Var}_{[0,1]}(f) \le K,$$

whereas for $\mathbb{L}_\infty$-approximation by piecewise constants with N *equidistant* knots the rate $\mathcal{O}(N^{-1})$ is attained only for $\mathbb{W}_\infty^1$, roughly the class of continuously differentiable functions, which is much narrower than the class of functions of bounded variation.

It was R. DeVore who had much advocated the studies of the non-linear methods in k-monotone approximation. Set

$$E_{N,r}(f)_p := E(f,\mathcal{S}_{N,r})_p\,, \quad E_{N,r}^{(k)}(f)_p := E^{(k)}(f,\mathcal{S}_{N,r})_p\,.$$

Notice that since $\mathcal{M}^k(0,1) \subset \mathbb{C}^{k-2}(0,1)$ (see Lemma 3.1), the set $\mathcal{S}_{N,r} \cap \mathcal{M}^k$ contains functions other than k-monotone polynomials of order r only if $r \ge k$. In 1995, Leviatan & Shadrin [8] and Petrov [13], independently, proved that for $k=1,2$ $r \ge k$, and $0 < p \le \infty$, there exists a constant $c_0 = \mathcal{O}(r)$ such that, for any $f \in \mathcal{M}_p^k$, $k=1,2$,

$$E_{c_0 N,r}^{(k)}(f)_p \le E_{N,r}(f)_p\,. \tag{1.2}$$

This result showed that the order of monotone and convex approximation by free knot splines is essentially the same as that in the non-constrained case, which, in view of (1.1), is a striking contrast to the linear approximation methods. Naturally, one would expect that the situation is similar for $k \ge 3$. However, the technique used in [8], [13] was based on some explicit constructions and some properties of monotone and convex functions which have no straightforward analogues for general k. (Say, for $k=1,2$ the maximum of two k-monotone functions is a k-monotone function, while this is no longer true for larger k.) Petrov [14] has managed to adopt this technique for $k=3$ and $p=\infty$ obtaining an analogue of (1.2), but it became clear that, for general $k \in \mathbb{N}$, new ideas are required.

Here, we prove the following general result.

Theorem 1.1 *Let $k,r,N \in \mathbb{N}$, $r \ge k$, and $0 < p \le \infty$. Then, there exist constants $c_0 \le C(k)\max(1,r-k)$ and $c_1 = c_1(r,k,p)$ such that, for all $f \in \mathcal{M}_p^k$,*

$$E_{c_0 N,r}^{(k)}(f)_p \le c_1 E_{N,r}(f)_p\,. \tag{1.3}$$

Using [8, Lemma 3], the following result on k-monotone approximation by smooth splines is an immediate corollary of Theorem 1.1.

Corollary 1.2 *Let $k, r, N \in \mathbb{N}$, $r \geq k$, and $0 < p \leq \infty$, and denote $\widetilde{E}^{(k)}_{N,r}(f)_p := E^{(k)}(f, \mathcal{S}_{N,r} \cap \mathbb{C}^{(r-2)})_p$. Then, there exist constants $c_0 \leq C(k) \max(1, r-k)$ and $c_1 = c_1(r,k,p)$ such that, for all $f \in \mathcal{M}^k_p$,*

$$\widetilde{E}^{(k)}_{c_0 N,r}(f)_p \leq c_1 E_{N,r}(f)_p \, .$$

For $k = 1$ and 2, Theorem 1.1 is an immediate consequence of (1.2). Because functions in $\mathcal{M}^1(a,b)$ (unlike those in $\mathcal{M}^k(a,b)$ with $k \geq 2$) do not have to be continuous everywhere on (a,b), the case $k = 1$ is somewhat different from $k \geq 2$ (though constructions are much simpler and some auxiliary statements become trivial if one lets k be equal to 1). Thus, in order to make this paper more readable, we concentrate below only on the more difficult case $k \geq 2$. At the same time, we mention that some of the statements are valid or can be modified to become valid for $k = 1$ as well.

Now, all direct results for the best (unconstrained) free knot spline approximation are being readily extended for the k-monotone case.

Corollary 1.3 *Let $k, r, N \in \mathbb{N}$, $r \geq k$, and let $f \in \mathcal{M}^k_\infty$ be such that $f^{(r-1)}$ is of bounded variation on $[0,1]$. Then,*

$$E^{(k)}_{N,r}(f)_\infty \leq c(r,k) N^{-r} \operatorname{Var}_{[0,1]}(f^{(r-1)}) \, .$$

This corollary is an immediate consequence of Theorem 1.1 and [4, Theorem 12.4.5]. It is related to an earlier result of Hu [5] which was actually the first result in k-monotone approximation by free knot splines: *For $f \in \mathbb{W}^r_1 \cap \mathcal{M}^k_\infty$, the order of k-monotone approximation by $S_{N,r}$ in $\mathbb{L}_\infty$ is $\mathcal{O}(N^{-r})$.*

The following corollary follows from Petrushev's estimate of (unconstrained) free knot spline approximation (see [16, Theorem 7.3] and [4, Theorem 12.8.2]).

Corollary 1.4 *Let $k, r, N \in \mathbb{N}$, $r \geq k$, $0 < p < \infty$, and $0 < \alpha < r$. Then, if $f \in \mathcal{M}^k_p \cap B^\alpha$,*

$$E^{(k)}_{N,r}(f)_p \leq c(\alpha, p, r) N^{-\alpha} |f|_{B^\alpha} \, ,$$

where $B^\alpha := B^\alpha_\gamma(L_\gamma)$, $1/\gamma = \alpha + 1/p$, denotes the Besov space with the semi-norm $|f|_{B^\alpha}$ defined by

$$|f|_{B^\alpha} = \left(\int_0^\infty t^{-\alpha\gamma - 1} \omega_r(f,t)^\gamma_\gamma \, dt \right)^{1/\gamma} .$$

Let us comment on the constants c_0, c_1 involved in (1.3), namely on the question, whether it is possible to have any (or both) of them equal to 1.

Leviatan & Shadrin [8] showed that, in order to retain the same degree of approximation for the k-monotone free knot splines approximation as for the best one, the increase of the knot number is unavoidable if $r \geq k+2$. Precisely, *for any $r \geq k+2$, $N \in \mathbb{N}$, $0 < p \leq \infty$, any $c > 0$, and $c_* = 2\lfloor \frac{r-k}{2} \rfloor$, there exists a function $f \in \mathcal{M}^k_\infty$ such that*

$$E^{(k)}_{c_* N,r}(f)_p > c E_{N,r}(f)_p, \quad r \geq k+2.$$

Thus, the question about whether or not it is necessary to increase the number of knots remains open only for $r = k$ and $k+1$.

On the other hand, for $r = k$ and $p = \infty$, a part of a theorem by Johnson (see Braess [2, Theorem VIII.3.4, p. 238]) is that *for any k, the best free knot spline approximant of order k to a k-monotone function in the $\mathbb{L}_\infty$-norm is k-monotone itself*, *i.e.,* in this case, $c_0 = c_1 = 1$, $r = k$, and, for any $f \in \mathcal{M}^k_\infty$,

$$E^{(k)}_{N,k}(f)_\infty = E_{N,k}(f)_\infty \, . \tag{1.4}$$

It would be interesting to find the exact order of $c_0(r,k)$ as a function of r and k. Estimates (1.2) and (1.4) also suggest another question; namely, whether the value $c_1 = 1$ in (1.3) can be attained with some $c'_0 = c'_0(r,k)$.

Notations. We let $I = (a,b)$ if not stated otherwise, and set $\mathbb{L}_p := \mathbb{L}_p(I)$, $\| \cdot \|_p := \| \cdot \|_{\mathbb{L}_p(I)}$, $\mathcal{S}_{N,k} := \mathcal{S}_{N,k}(I)$, etc., *i.e.,* the interval I is omitted if there is no risk of confusion.

Further, $f^{(i)}(x+)$ and $f^{(i)}(x-)$ denote the right and the left i-th derivatives of f at x, respectively.

Notations $c_{p,r,k}$ and $c(p,r,k)$ stand for a constant which depends only on the parameters given (p, r, and k in this case), where, for $0 < p \le \infty$, dependence on p means dependence on $\min(1,p)$.

The "prime"-notation k' is going to be reserved for $\lfloor k/2 \rfloor + 1$ throughout this paper:

$$k' := \lfloor k/2 \rfloor + 1\,.$$

For $f \in \mathbb{L}_p(a,b)$ and a set $\mathcal{U} \subset \mathbb{L}_p(a,b)$, we define

$$\mathcal{P}_{\mathcal{U}}(f)_p := \mathcal{P}_{\mathcal{U}}(f)_{\mathbb{L}_p(a,b)} := \{u \in \mathcal{U} : \ \|f-u\|_p = E(f,\mathcal{U})_p\}.$$

In other words, $\mathcal{P}_{\mathcal{U}}(f)_p$ is the set of all best $\mathbb{L}_p$-approximants to f from $\mathcal{U}$ on (a,b).

2 Outline of the proof

The general direction of the proof is the same as it was for $k = 1, 2$: given a k-monotone function f, one takes $\sigma \in \mathcal{P}_{S_{N,r}}(f)_p$, a best free knot spline approximant to f (which is not necessarily k-monotone) and puts some corrections in it trying to convert it into a k-monotone spline preserving the approximation order. For $k = 1$ and 2, these corrections were done by explicit constructions which, unfortunately, have no straightforward generalizations for $k \ge 3$, and so our basic idea came from the following general considerations.

There is another notion of k-monotone approximation in which a function f which is not in $\mathcal{M}^k$ is being approximated by elements from the entire $\mathcal{M}^k$ ($\mathcal{M}^k$ is a convex cone). There is an extended literature on this subject where one studies existence and uniqueness of this type best k-monotone approximant, its characterization and structural properties, see e.g. [18] and the references therein. When can one have a need to approximate an arbitrary function by a k-monotone one? The only situation we can think of is the necessity to correct the data which must be k-monotone by some a priori assumptions. This is exactly the case of shape preserving approximation, and this is how we correct σ.

Given $f \in \mathcal{M}^k$, we take $\sigma \in \mathcal{P}_{\mathcal{S}_{N,r}}(f)_p$, a best free knot spline approximant to f, and correct σ by $\mathfrak{f}_* \in \mathcal{P}_{\mathcal{M}^k}(\sigma)_p$, a best approximant to σ from $\mathcal{M}^k$.

Here are two observations concerning this idea.

1) *Approximation property of* $\mathfrak{f}_*$. The function f belongs to $\mathcal{M}^k$, but $\mathfrak{f}_*$ is a best approximant to σ from $\mathcal{M}^k$, hence

$$\|\sigma - \mathfrak{f}_*\|_p \le \|\sigma - f\|_p\,.$$

Therefore,

$$c_p\,\|f - \mathfrak{f}_*\|_p \le \|f-\sigma\|_p + \|\sigma - \mathfrak{f}_*\|_p \le 2\|f-\sigma\|_p$$

i.e., $\mathfrak{f}_*$ approximates f as well as σ.

2) *Spline structure of* $\mathfrak{f}_*$. A result from the theory of approximation by elements of $\mathcal{M}^k$ reads that (in the "piecewise sense") either $\mathfrak{f}_*$ is identical with σ (which is a spline of order r) or it is a spline of order k (because the functions $g(x) = \sum_\alpha c_\alpha (x-x_\alpha)_+^{k-1}$, $c_\alpha > 0$, are the boundary points of the cone $\mathcal{M}^k$). Thus, $\mathfrak{f}_*$ is a spline of order r. If $\mathfrak{f}_*$ had $\mathcal{O}(N)$ knots, then we could stop at this point. The problem is that it may have too many knots (infinitely many, in fact).

The paper is organized as follows.

1) First of all, to ease the exposition, we switch to a local version of the idea described above, and correct separately each polynomial part of σ by its best approximation $\mathfrak{f}_*$ from $\mathcal{M}^k[f]$, a subclass of k-monotone functions defined locally (see §3 for precise definition of $\mathcal{M}^k[f]$).

2) In §3, we cite some known results concerning existence and structure of the elements $\mathfrak{f}_* \in \mathcal{P}_{\mathcal{M}^k[f]}(\sigma)_p$. As mentioned earlier, $\mathfrak{f}_*$ is a spline of order r, but it may have too many knots to be in $\mathcal{S}_{cN,r}$, in which case we modify it into an appropriate spline s.

3) Properties of s are formulated as Proposition 4.2 in §4 where we use them to prove Theorem 1.1.

4) The proof of Proposition 4.2 takes the rest of the paper. In §§5–7, we blend $\mathfrak{f}_*$ with the polynomial parts of σ using some results from the theory of moments, and consider some general aspects of this procedure. In §8, we prepare to show that the blending spline s approximates f as well as $\mathfrak{f}_*$, and the final §9 joins all the parts of the proof together.

Remark 2.1 The number of knots of $\mathfrak{f}_* \in \mathcal{P}_{\mathcal{M}^k[f]}(\sigma)_p$ is approximately the same as the number of distinct zeros of $\sigma - \mathfrak{f}_*$ (see Lemma 3.8 below). In our proofs, we assume that this number may be arbitrarily large. However, we conjecture that this is not the case, *i.e.,* a best k-monotone approximant to a piecewise polynomial σ (and perhaps to any piecewise k-monotone function) with M pieces has only $\mathcal{O}(M)$ points of intersection with σ. If it is so, then there is no need in considerations given in §§5–9. This conjecture is true for $k = 1, 2$ as one can easily check, and our method gives a simpler proof for these cases than in [8] and [13]. For $k \geq 3$, the problem is open.

Remark 2.2 Actually, the correction of σ made explicitly for $k = 1, 2$ in [8] and [13], is exactly the best k-monotone approximation of σ from $\mathcal{M}^k[f]$ under additional restriction that this is also one-sided approximation. This restriction provides the constant $c_1 = 1$ on the right-hand side of (1.2). For $k \geq 3$ we cannot pose such a restriction, hence $c_1 > 1$ in (1.3).

3 Classes $\mathcal{M}^k[f]$ and their Properties

The following lemma lists some basic properties of k-monotone functions for $k \geq 2$.

Lemma 3.1 *The following statements are equivalent for* $k \geq 2$*:*

(0) $f \in \mathcal{M}^k(0, 1)$.

(1) $f^{(k-2)}$ *exists and is convex on* $(0, 1)$.

(2) $f^{(k-2)}$ *is absolutely continuous on any closed subinterval of* $(0, 1)$*, and has left and right derivatives,* $f^{(k-1)}(\cdot-)$ *and* $f^{(k-1)}(\cdot+)$*, which are, respectively, left- and right-continuous and nondecreasing on* $(0, 1)$.

(3) *For each closed subinterval* $[a, b] \subset (0, 1)$*, there is a polynomial* $p \in \Pi_k$ *and a bounded nondecreasing function* μ *such that*

$$f(x) = p(x) + \frac{1}{k!} \int_a^b k(x - t)_+^{k-1} d\mu(t), \quad x \in [a, b].$$

Proof. See Bullen [3, Theorem 7, Corollary 8]. See also [12], [17] for various properties of k-monotone functions (called there “k-convex”) and their applications. ∎

Lemma 3.1(2) allows us to introduce the following classes of function.

By $\mathcal{M}^k_{a+} := \mathcal{M}^k_{a+}(a, b)$ and $\mathcal{M}^k_{b-} := \mathcal{M}^k_{b-}(a, b)$ we denote the subclasses of those functions $f \in \mathcal{M}^k(a, b)$ for which the values $\{f^{(i)}(a+)\}_{i=0}^{k-1}$ and $\{f^{(i)}(b-)\}_{i=0}^{k-1}$, respectively, are finite, and set $\mathcal{M}^k_* := \mathcal{M}^k_*(a, b) := \mathcal{M}^k_{a+} \cap \mathcal{M}^k_{b-}$.

For $f \in \mathcal{M}^k_{a+}$ and $g \in \mathcal{M}^k_{b-}$, we define

$$\mathcal{M}^k_{a+}[f] := \left\{h \in \mathcal{M}^k \,\middle|\, h^{(i)}(a+) = f^{(i)}(a+),\ i = 0, \ldots, k-2; \quad h^{(k-1)}(a+) \geq f^{(k-1)}(a+)\right\},$$

$$\mathcal{M}^k_{b-}[g] := \left\{h \in \mathcal{M}^k \,\middle|\, h^{(i)}(b-) = g^{(i)}(b-),\ i = 0, \ldots, k-2; \quad h^{(k-1)}(b-) \leq g^{(k-1)}(b-)\right\}.$$

Finally, let

$$\mathcal{M}^k[f, g] = \mathcal{M}^k_{a+}[f] \cap \mathcal{M}^k_{b-}[g],$$

and, for $f \in \mathcal{M}^k_*$,

$$\mathcal{M}^k[f] = \mathcal{M}^k[f, f].$$

Note that $\mathcal{M}^k[f]$ is always nonempty (it contains f), while $\mathcal{M}^k[f, g]$ can be the empty set. In §7, we give a sufficient condition on f and g which guarantees that there is a function h from $\mathcal{M}^k[f, g]$.

Lemma 3.2 *Let* $f, g \in \mathcal{M}^k(0, 1)$*, and let* $[a, b] \subset (0, 1)$*. Then* $f, g \in \mathcal{M}^k_*(a, b)$*, and for any* $h \in \mathcal{M}^k[f, g](a, b)$ *(if it exists) the function*

$$\widetilde{h}(x) := \begin{cases} f(x), & x \in (0, a], \\ h(x), & x \in (a, b), \\ g(x), & x \in [b, 1), \end{cases}$$

belongs to $\mathcal{M}^k(0,1)$.

Proof. The proof is an immediate consequence of Lemma 3.1(2). ■

We will use Lemma 3.2 without further reference to build k-monotone functions from k-monotone pieces. For example, if $f \in \mathcal{M}^k_*(0,1)$, $\cup I_\ell = (0,1)$ with $I_\ell \cap I_{\ell'} = \emptyset$ if $\ell \neq \ell'$, and $h_\ell \in \mathcal{M}^k[f](I_\ell)$, then the function h, defined as $h := h_\ell$ on I_ℓ, belongs to $\mathcal{M}^k[f](0,1)$.

Now we consider some properties of approximation from $\mathcal{M}^k[f]$.

Lemma 3.3 *Let $k \geq 2$, $0 < p \leq \infty$ and $f \in \mathcal{M}^k_*(a,b)$. Then, for any $g \in \mathbb{L}_p$, an element of its best $\mathbb{L}_p$-approximation from $\mathcal{M}^k[f]$ exists,* i.e., *the set $\mathcal{P}_{\mathcal{M}^k[f]}(g)_p$ is not empty.*

Proof. The proof is based on the arguments similar to those used by Zwick [19, Theorem 4] for the case $p = \infty$. We give it here for completeness. Set

$$\alpha_i := f^{(i)}(a+) \quad \text{and} \quad \beta_i := f^{(i)}(b-)\,, \quad i = 0, \ldots, k-1\,,$$

and consider a sequence $(f_j) \subset \mathcal{M}^k[f]$ such that, for $j \in \mathbb{N}$,

$$\|f_j - g\|_p^p \leq E(g, \mathcal{M}^k[f])_p^p + 1/j, \quad \text{if} \quad 0 < p < 1\,,$$

and

$$\|f_j - g\|_p \leq E(g, \mathcal{M}^k[f])_p + 1/j, \quad \text{if} \quad 1 \leq p \leq \infty\,.$$

Since $f_j^{(k-2)}(x) = \alpha_{k-2} + \int_a^x f_j^{(k-1)}(t)\,dt$ and $\|f_j^{(k-1)}\|_\infty \leq \max\{|\alpha_{k-1}|, |\beta_{k-1}|\}$, we conclude that $(f_j^{(k-2)})$ is uniformly bounded and equicontinuous on $[a,b]$. Therefore, there exists a subsequence $(f_{j_s}^{(k-2)})$ which converges to a function h_* uniformly on $[a,b]$, and this h_* is necessarily convex and satisfies $h_*'(a+) \geq \alpha_{k-1}$ and $h_*'(b-) \leq \beta_{k-1}$. Now, the function $\mathfrak{f}_*$ such that $\mathfrak{f}_* := h_*$, if $k = 2$, and

$$\mathfrak{f}_*(x) := \sum_{i=0}^{k-3} \frac{\alpha_i}{i!}(x-a)^i + \frac{1}{(k-3)!}\int_a^b (x-t)_+^{k-3} h_*(t)\,dt\,, \quad k \geq 3,$$

is in $\mathcal{M}^k[f]$ and satisfies $\|g - \mathfrak{f}_*\|_p = E(g, \mathcal{M}^k[f])_p$, *i.e.*, $\mathfrak{f}_* \in \mathcal{P}_{\mathcal{M}^k[f]}(g)_p$. ■

Lemma 3.4 (Zwick [20]) *Let $k \in \mathbb{N}$ and $f \in \mathcal{M}^k_*(a,b)$. Then, there exist two splines $z_\nu = z_\nu(f, [a,b])$, $\nu = 1, 2$, such that*

$$z_1, z_2 \in \mathcal{M}^k[f] \cap \mathcal{S}_{k',k}, \quad k' = \lfloor k/2 \rfloor + 1,$$

and

$$z_1 \leq f \leq z_2 \quad \text{on} \quad [a,b]\,.$$

If f does not belong to $\mathcal{S}_{k',k}$, then the inequalities are strict, respectively, on some nonempty intervals I_1, I_2 in $[a,b]$.

Remark 3.5 In [20], more precise conclusions regarding the number of polynomial pieces k' of the splines z_ν and their boundary values are given. The proof is based on the Markov–Krein Theorem from the theory of moments.

Remark 3.6 We emphasize that k' denotes $\lfloor k/2 \rfloor + 1$ throughout this paper.

A simple, yet important, consequence of Lemma 3.4 is the following result on the structural properties of best $\mathbb{L}_p$-approximants from $\mathcal{M}^k[f]$.

Lemma 3.7 *For $k \geq 2$, $0 < p < \infty$, $I = (a,b)$, let $g \in \mathbb{C}$, $f \in \mathcal{M}^k_*$, and $\mathfrak{f}_* \in \mathcal{P}_{\mathcal{M}^k[f]}(g)_p$. If the difference $g - \mathfrak{f}_*$ has no zeros inside an interval $(c,d) \subset (a,b)$, then $\mathfrak{f}_* \in \mathcal{S}_{k',k}[c,d]$.*

Proof. The idea of the proof is similar to what was considered by Zwick [20] in the case $p = 1$. Suppose that $0 < p < \infty$. Without loss of generality we can assume that $\mathfrak{f}_*(x) > g(x)$, $x \in (c, d)$. Now, suppose that $\mathfrak{f}_* \notin \mathcal{S}_{k',k}[c+\epsilon, d-\epsilon]$ for some $\epsilon > 0$. Consider a function $\tilde{f}$ obtained from $\mathfrak{f}_*$ by replacing it on the interval $[c+\epsilon, d-\epsilon]$ by $z_1(\mathfrak{f}_*, [c+\epsilon, d-\epsilon])$ (see Lemma 3.4). Then, $\tilde{f} \in \mathcal{M}^k[f]$ and $\mathfrak{f}_* - \tilde{f} \geq 0$ on $[a, b]$ with this inequality being strict on a nonempty interval contained in $(c+\epsilon, d-\epsilon)$.

Since $\mathfrak{f}_* - g$ is a continuous positive function on a closed interval $[c+\epsilon, d-\epsilon]$, there exists $\delta > 0$ such that $\mathfrak{f}_*(x) \geq g(x) + \delta$, $x \in [c+\epsilon, d-\epsilon]$. Therefore, there exists $0 < \mu < 1$ such that $\widehat{f}(x) := \mu\mathfrak{f}_*(x) + (1-\mu)\tilde{f}(x)$ satisfies the inequalities $g(x) < \widehat{f} \leq \mathfrak{f}_*$ on $[c+\epsilon, d-\epsilon]$, and $\|\mathfrak{f}_* - \widehat{f}\|_{\mathbb{L}_p[c+\epsilon,d-\epsilon]} \neq 0$. This implies that $\|\widehat{f} - g\|_p < \|\mathfrak{f}_* - g\|_p$ which contradicts our assumption that $\mathfrak{f}_* \in \mathcal{P}_{\mathcal{M}^k[f]}(g)_p$.

Hence, $\mathfrak{f}_* \in \mathcal{S}_{k',k}[c+\epsilon, d-\epsilon]$ for all $\epsilon > 0$, which implies that $\mathfrak{f}_* \in \mathcal{S}_{k',k}[c, d]$. ∎

Lemma 3.8 *For $k \geq 2$, $0 < p < \infty$, $I = (a, b)$, let $g \in \mathbb{C}$, $f \in \mathcal{M}^k_*$, and $\mathfrak{f}_* \in \mathcal{P}_{\mathcal{M}^k[f]}(g)_p$. Further, let $\mathfrak{Z}$ be the set of zeros of $g - \mathfrak{f}_*$,* i.e.,

$$\mathfrak{Z} := \{z \in I \,\big|\, g(z) = \mathfrak{f}_*(z)\},$$

and let $\mathfrak{Z}^*$ be the set of all limit points of $\mathfrak{Z}$. Then, the following is true.

(1) *$\mathfrak{f}_* = g$ on $\mathfrak{Z}^*$.*

(2) *If, for a closed interval $[c, d] \subset I \setminus \mathfrak{Z}^*$, the difference $g - \mathfrak{f}_*$ has (necessarily finitely many) $m-1$ distinct zeros in (c, d), then $\mathfrak{f}_* \in \mathcal{S}_{mk',k}[c, d]$.*

Proof. This lemma is a variation of Zwick [20, Theorem 2]. In a similar form (though with $\mathfrak{Z}^*$ defined differently), it appeared in Marano [11]. Part 1 immediately follows from continuity of g and $\mathfrak{f}_*$. Part 2 is a consequence of Lemma 3.7. ∎

For $p = \infty$, Lemma 3.7 is not valid, because local changes influence the integral's value, but not necessarily the sup-norm, hence there may be best k-monotone $\mathbb{L}_\infty$-approximants with the structure different from that specified in Lemma 3.8. However, for our purposes, it is enough that there is at least one element from $\mathcal{P}_{\mathcal{M}^k[f]}(g)_\infty$ that has the spline structure. The following statement is valid.

Lemma 3.9 *For $k \geq 2$, $p = \infty$, $I = (a, b)$, let $g \in \mathbb{C}$ and $f \in \mathcal{M}^k_*$. Then, there exists $\mathfrak{f}_* \in \mathcal{P}_{\mathcal{M}^k[f]}(g)_\infty$ such that all the conclusions of Lemma 3.8 hold true.*

Proof. The idea of the proof is to take as $\mathfrak{f}_*$ an element which minimizes, say, the L_2-norm of $g - \mathfrak{f}_*$ over $\mathfrak{f}_* \in \mathcal{P}_{\mathcal{M}^k[f]}(g)_\infty$. We omit details. ∎

Now the spline structure of the best k-monotone approximant to any spline readily follows.

Corollary 3.10 *For $r \geq k \geq 2$, $0 < p \leq \infty$, $I = (a, b)$, let $g \in \mathcal{S}_{N,r} \cap \mathbb{C}$ and $f \in \mathcal{M}^k_*$. Then, there is a $\mathfrak{f}_* \in \mathcal{P}_{\mathcal{M}^k[f]}(g)_p$ which is a piecewise polynomial of order r.*

4 Proof of Theorem 1.1

The following three propositions are the main components of the proof.

Proposition 4.1 *For $k, r \in \mathbb{N}$, $r \geq k \geq 2$, $0 < p \leq \infty$, $I = (a, b)$, let $f \in \mathcal{M}^k_*$ and $(-\mathfrak{p}) \in (\Pi_r \setminus \Pi_k) \cap \mathcal{M}^k$. Then there exists a spline s such that*

$$s \in \mathcal{S}_{(k+1)k',k} \cap \mathcal{M}^k[f]$$

and

$$\|\mathfrak{p} - s\|_p = E(\mathfrak{p}, \mathcal{M}^k[f])_p.$$

Proof. Let us show that $\mathfrak{f}_*$, a best approximant to $\mathfrak{p}$ from $\mathcal{M}^k[f]$, satisfies all the conclusions of the proposition (hence, $s := \mathfrak{f}_*$). Since, by the definition,

$$\mathfrak{f}_* \in \mathcal{M}^k[f], \quad \|\mathfrak{p} - \mathfrak{f}_*\|_p = E(\mathfrak{p}, \mathcal{M}^k[f])_p,$$

only the spline structure needs to be proved. Since $(-\mathfrak{p})$ is a k-monotone polynomial of degree $> k-1$, it is a strictly k-monotone function in the sense that $(-\mathfrak{p})^{(k-2)}$ is strictly convex. Hence the function $(\mathfrak{f}_* - \mathfrak{p})^{(k-2)}$ is strictly convex too, thus it has at most two zeros, and, therefore, $\mathfrak{f}_* - \mathfrak{p}$ has not more than k distinct zeros on I. By Lemma 3.8 (or Lemma 3.9 in the case $p = \infty$), $\mathfrak{f}_* \in \mathcal{S}_{(k+1)k',k}$, and the proof is complete. ■

Proposition 4.2 *Let $k, r \in \mathbb{N}$, $r \geq k \geq 2$, $0 < p \leq \infty$, $I = (a, b)$, $f \in \mathcal{M}^k_*$, and $\mathfrak{p} \in \Pi_r \cap \mathcal{M}^k$. Then there exist a constant $C(k)$ independent of I and a spline $s \in \mathcal{S}_{C(k),r} \cap \mathcal{M}^k[f]$ such that*

$$\|\mathfrak{p} - s\|_p \leq c_2 E(\mathfrak{p}, \mathcal{M}^k[f])_p, \quad c_2 = c_2(p, r, k)\,.$$

Now, $\mathfrak{f}_*$ from $\mathcal{P}_{\mathcal{M}^k[f]}(\mathfrak{p})_p$ is still a piecewise polynomial of order r, but we cannot take $s = \mathfrak{f}_*$ because two k-monotone functions ($\mathfrak{f}_*$ and $\mathfrak{p}$ in our case) may have any number of intersections, hence $\mathfrak{f}_*$ may have any number of knots. We obtain s as a modification of $\mathfrak{f}_*$, which will be done in the following sections with the proof of Proposition 4.2 given in §9.

Proposition 4.3 *Let $k, r \in \mathbb{N}$, $r \geq k \geq 2$, $0 < p \leq \infty$, $I = (a, b)$, $f \in \mathcal{M}^k_*$, and let $\mathfrak{p}$ be such that either $\mathfrak{p} \in \Pi_r \cap \mathcal{M}^k$ or $(-\mathfrak{p}) \in (\Pi_r \setminus \Pi_k) \cap \mathcal{M}^k$. Then there exists a spline s such that*

$$s \in \mathcal{S}_{C(k),r} \cap \mathcal{M}^k[f]$$

and

$$\|f - s\|_p \leq c_1 \, \|f - \mathfrak{p}\|_p \,, \quad c_1 = c_1(p, r, k)\,. \tag{4.1}$$

Proof. Let s be the spline from either of Propositions 4.1 and 4.2, so that $s \in \mathcal{S}_{C(k),r} \cap \mathcal{M}^k[f]$ and

$$\|\mathfrak{p} - s\|_p \leq c_2 E(\mathfrak{p}, \mathcal{M}^k[f])_p. \tag{4.2}$$

We only need to prove (4.1). Using the triangle inequality, and the estimate (4.2) we obtain

$$c_p \|f - s\|_p \leq \|f - \mathfrak{p}\|_p + \|\mathfrak{p} - s\|_p \leq \|f - \mathfrak{p}\|_p + c_2 \, E(\mathfrak{p}, \mathcal{M}^k[f])_p.$$

Since f belongs to $\mathcal{M}^k[f]$ in a trivial manner, it follows that

$$E(\mathfrak{p}, \mathcal{M}^k[f])_p := \inf_{u \in \mathcal{M}^k[f]} \|\mathfrak{p} - u\|_p \leq \|\mathfrak{p} - f\|_p.$$

Thus

$$c_p \|f - s\|_p \leq (c_2 + 1) \|f - \mathfrak{p}\|_p\,.$$

■

Finally, the following lemma shows that, in the proof of Theorem 1.1, instead of an arbitrary $f \in \mathcal{M}^k_p(0,1)$, we may consider $f \in \mathcal{M}^k_*(0,1)$, *i.e.,* we may assume that the function f and its derivatives are bounded at the endpoints.

Lemma 4.4 *Let $k \in \mathbb{N}$, $0 < p \leq \infty$, and $f \in \mathcal{M}^k_p(0,1)$. Then, for any $\epsilon > 0$, there exists $f_\epsilon \in \mathcal{M}^k_*(0,1)$ such that*

$$\|f - f_\epsilon\|_p < \epsilon.$$

Proof. For $f \in \mathcal{M}_p^k(0,1)$ and $x_0 \in (0,1)$, let T_{x_0} be the Taylor polynomial of degree $k-1$ at x_0+ (or at x_0-), *i.e.*,

$$T_{x_0}(x) := \sum_{i=0}^{k-1} \tfrac{1}{i!} f^{(i)}(x_0+)(x-x_0)^i.$$

Given ϵ, for δ to be prescribed, let

$$f_\epsilon := \begin{cases} T_\delta, & \text{on} \quad [0,\delta], \\ f, & \text{on} \quad [\delta, 1-\delta], \\ T_{1-\delta}, & \text{on} \quad [1-\delta, 1]. \end{cases}$$

Then obviously $f_\epsilon \in \mathcal{M}_*^k(0,1)$ and

$$\|f - f_\epsilon\|_p \le c_p \|f - T_\delta\|_{\mathbb{L}_p[0,\delta]} + c_p \|f - T_{1-\delta}\|_{\mathbb{L}_p[1-\delta,1]}. \tag{4.3}$$

¿From [6, Theorem 1], it follows that, for $I = (a,b)$, $f \in \mathcal{M}_p^k(I)$, and $x_* := \frac{a+b}{2}$, we have

$$\|f - T_{x_*}\|_{\mathbb{L}_p(I)} \le c_{k,p}\, \omega_k(f)_{\mathbb{L}_p(I)},$$

where $\omega_k(f)_{\mathbb{L}_p(I)}$ is the k-th modulus of smoothness of $f \in \mathbb{L}_p(I)$ (see §8 for the definition), which, as is well known, has the property that $\omega_k(f)_{\mathbb{L}_p(J)} \to 0$ if $|J| \to 0$, $J \subset I$. Applying this result to the interval $(0, 2\delta) \subset (0,1)$ we obtain

$$\|f - T_\delta\|_{\mathbb{L}_p(0,\delta)} \le \|f - T_\delta\|_{\mathbb{L}_p(0,2\delta)} \le c_{k,p}\, \omega_k(f)_{\mathbb{L}_p(0,2\delta)} \to 0 \quad \text{as} \quad \delta \to 0\,.$$

Similarly,

$$\|f - T_{1-\delta}\|_{\mathbb{L}_p(1-\delta,1)} \le c_{k,p}\, \omega_k(f)_{\mathbb{L}_p(1-2\delta,1)} \to 0 \quad \text{as} \quad \delta \to 0\,.$$

■

Proof of Theorem 1.1. By Lemma 4.4, we can assume that $f \in \mathcal{M}_*^k(0,1)$. Let $\sigma \in \mathcal{S}_{N,r}$ be a spline of best $\mathbb{L}_p$-approximation to f on $(0,1)$. We need to prove that there exists a spline s such that

$$s \in \mathcal{S}_{c_0 N, r} \cap \mathcal{M}^k(0,1) \quad \text{and} \quad \|f - s\|_p \le c_1 \|f - \sigma\|_p\,.$$

Denote by $\{J_m\}$ the set of largest subintervals of $[0,1]$ on which σ is a polynomial of order r, and by $\{I_\ell\}$ the set of largest subintervals of J_m's on which $\sigma^{(k)}$ has a constant sign. Since $\sigma \in \mathcal{S}_{N,r}[0,1]$, there are at most N intervals J_m, and, on each J_m, the spline $\sigma^{(k)}$ is a polynomial of degree $r-1-k$, hence there are at most $\max(1, r-k)$ subintervals I_ℓ in each interval J_m. Thus, $\{I_\ell\}$ is a partition of $[0,1]$ such that

$$[0,1] = \cup I_\ell, \quad \#\{I_\ell\} \le N \max(1, r-k)\,,$$

and, on each I_ℓ,

$$\text{either} \quad \sigma \in \Pi_r \cap \mathcal{M}^k, \qquad \text{or} \quad (-\sigma) \in (\Pi_r \setminus \Pi_k) \cap \mathcal{M}^k\,.$$

By Proposition 4.3, on each interval I_ℓ, there exists a spline s_ℓ such that

$$s_\ell \in \mathcal{S}_{C(k),r} \cap \mathcal{M}^k[f](I_\ell)$$

and

$$\|f - s_\ell\|_{\mathbb{L}_p(I_\ell)} \le c_1 \|f - \sigma\|_{\mathbb{L}_p(I_\ell)}\,. \tag{4.4}$$

Now, define the spline s so that

$$s := s_\ell \quad \text{on} \quad I_\ell\,.$$

Relations $s_\ell \in \mathcal{S}_{C(k),r}(I_\ell)$ and $\#\{I_\ell\} \le N \max(1, r-k)$ imply that

$$s \in \mathcal{S}_{c_0 N, r}(0,1), \quad c_0 = C(k) \max(1, r-k),$$

while inclusions $s_\ell \in \mathcal{M}^k[f](I_\ell)$ with $\cup I_\ell = [0,1]$ yield

$$s \in \mathcal{M}^k[f](0,1) \subset \mathcal{M}^k(0,1)\,.$$

Thus,

$$s \in \mathcal{S}_{c_0 N,r} \cap \mathcal{M}^k(0,1).$$

Finally, to estimate the degree of approximation of f by s for $0 < p < \infty$ (modifications for $p = \infty$ are obvious), from (4.4) we obtain

$$\|f - s\|^p_{\mathbb{L}_p(0,1)} \le \sum_\ell \|f - s_\ell\|^p_{\mathbb{L}_p(I_\ell)} \le c_1^p \sum_\ell \|f - \sigma\|^p_{\mathbb{L}_p(I_\ell)} = c_1^p \|f - \sigma\|^p_{\mathbb{L}_p(0,1)}$$

i.e.,

$$E^{(k)}_{c_0 N,r}(f)_p \le c_1 E_{N,r}(f)_p$$

■

5 k-monotone interpolation

If $\mathfrak{p} - \mathfrak{f}_*$ has many intersections (see Proposition 4.2), then the spline $\mathfrak{f}_* \in \mathcal{P}_{\mathcal{M}^k[f]}(\mathfrak{p})$ has many knots. In this case, we will modify $\mathfrak{f}_*$ into a spline s with a smaller number of knots by blending f_* with $\mathfrak{p}$. This procedure is related to the following general problem.

Problem 5.1 Given two k-monotone functions f, g on J, and an interval $(a,b) \subset J$, determine whether or not there exists a k-monotone function h in $\mathcal{M}^k[f,g](a,b)$. Note that existence of such h implies that there is a function $\widetilde{h}$ such that

$$\widetilde{h} \in \mathcal{M}^k(J), \quad \text{and} \quad \widetilde{h}(x) = \begin{cases} f(x), & x \le a, \\ g(x), & x \ge b. \end{cases}$$

We will refer to this problem as *blending* of $f, g \in \mathcal{M}^k(J)$ on $[a,b]$. Actually, all we need is a k-monotone interpolation of data $f^{(i)}(a+), g^{(i)}(b-)$, $i = 0, \dots, k-1$, so that we consider this topic more generally.

Let

$$\boldsymbol{x} := (x_i)_{i=1}^{n+k} := \{a = x_1 \le \ldots \le x_{n+k} = b\}$$

be a sequence of interpolation knots such that $x_i < x_{i+k}$, and let

$$\boldsymbol{y} := \boldsymbol{y}(\boldsymbol{x}) := (y_i)_{i=1}^{n+k}\,.$$

We use the usual convention that, if some of the knots in $\boldsymbol{x}$ are repeated then interpolation of corresponding derivatives takes place. For each $j = 1, \dots, n+k$, denote by l_j the number of points x_i such that $x_i = x_j$ with $i \le j$, *i.e.*,

$$l_j := l_j(\boldsymbol{x}) := \#\left\{i \;\middle|\; 1 \le i \le j,\; x_i = x_j\right\}\,.$$

Note that, because of the restriction $x_i \ne x_{i+k}$, the inequality $l_j \le k$ is valid for all j.

Definition 5.2 A data sequence $(\boldsymbol{x}, \boldsymbol{y}) := (x_i, y_i)_{i=1}^{n+k}$ is called k-monotone if there exists a k-monotone function $f \in \mathcal{M}^k_*(a,b)$ such that

$$f^{(l_j - 1)}(x_j) = y_j, \quad j = 1, \dots, n+k\,. \tag{5.1}$$

Note that if all the knots in $\boldsymbol{x}$ are distinct then the sequence $(\boldsymbol{x}, \boldsymbol{y})$ is k-monotone if $f(x_i) = y_i$, $j = 1, \ldots, n+k$, for some $f \in \mathcal{M}_*^k(a,b)$. Also, if $l_j = k$ for some j, then $f^{(l_j-1)}(x_j) = f^{(k-1)}(x_j)$ is understood as $f^{(k-1)}(x_j+)$ or $f^{(k-1)}(x_j-)$.

Since

$$f \in \mathcal{M}^k \quad \Leftrightarrow \quad [t_i, \ldots, t_{i+k}]f \geq 0 \quad \forall (t_i),$$

where not all t_i's are the same, one must necessarily have for a k-monotone sequence $(\boldsymbol{x}, \boldsymbol{y})$

$$[x_i, \ldots, x_{i+k}]\boldsymbol{y} \geq 0.$$

If $k = 1$ or 2 (*i.e.,* in the case of monotone or convex interpolation), this condition is sufficient as well. However, it is *not* sufficient if $k \geq 3$, as the following example shows.

Example 5.3 The data set

$\boldsymbol{x}$	$\boldsymbol{y}$	δ^1	δ^2	δ^3
-5	-77			
		25		
-3	-27		-3	
		13		0
-1	-1		-3	
		1		1
1	1		3	
		13		0
3	27		3	
		25		
5	77			

has nonnegative divided differences of order 3, but, at the same time,

$$\begin{aligned}
[-5,-3,-1,0]\boldsymbol{y} + [0,1,3,5]\boldsymbol{y} &= -\tfrac{1}{5}[-5,-3,-1]\boldsymbol{y} - \tfrac{1}{15}[-3,-1]\boldsymbol{y} - \tfrac{1}{15}[-1]\boldsymbol{y} + \tfrac{1}{15}[0]\boldsymbol{y} \\
&\qquad + \tfrac{1}{5}[1,3,5]\boldsymbol{y} \quad - \tfrac{1}{15}[1,3]\boldsymbol{y} \quad + \tfrac{1}{15}[1]\boldsymbol{y} - \tfrac{1}{15}[0]\boldsymbol{y} \\
&= \tfrac{1}{5}\cdot 6 - \tfrac{1}{15}\cdot 26 + \tfrac{1}{15}\cdot 2 = -\tfrac{6}{15} < 0.
\end{aligned}$$

Hence, there is no 3-monotone function passing through $(\boldsymbol{x}, \boldsymbol{y})$.

Denote by

$$\boldsymbol{v} := \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y}) := (v_i)_1^n, \quad v_i := [x_i, \ldots, x_{i+k}]\boldsymbol{y}\,,$$

the sequence of divided differences of $\boldsymbol{y}(\boldsymbol{x})$, and by

$$\mathfrak{M} := \mathfrak{M}(\boldsymbol{x}) := (\tfrac{1}{k!}M_i), \quad M_i(t) := k\,[x_i, \ldots, x_{i+k}](\cdot - t)_+^{k-1}\,,$$

the sequence of the B-splines of order k with the knot sequence $\boldsymbol{x}$. Recall that $\operatorname{supp} M_i = [x_i, x_{i+k}]$, $M_i \geq 0$, $\int M_i = 1$, and that, for any $f \in \mathbb{C}^k(a,b)$ (in fact, condition $f \in \mathbb{W}_1^k(a,b)$ is sufficient),

$$[x_i, \ldots, x_{i+k}]f = \frac{1}{k!}\int_a^b M_i(t) f^{(k)}(t)\,dt\,.$$

Notice that if a k-monotone function f belongs to $\mathbb{C}^k$, then $f^{(k)} \geq 0$. Thus, to check whether the data sequence (x_i, y_i) is k-monotone , one needs to form the sequence of divided differences (v_i) and check whether there is a non-negative function λ such that

$$v_i = \frac{1}{k!}\int M_i(t)\lambda(t)dt.$$

The last problem is the so-called *Markov moment problem* which we discuss in the next section.

6 Markov moment problem and k-monotone interpolation

Let $\mathcal{U} := (u_i)_{i=1}^n$ be a sequence of continuous linearly independent real-valued functions on $I = (a, b)$, and let $\boldsymbol{v} := (v_i)_{i=1}^n$ be a sequence of real numbers.

Definition 6.1 A sequence $\boldsymbol{v} \in \mathbb{R}^n$ is called a moment sequence w.r.t. $\mathcal{U}$ if, for some bounded non-decreasing function μ, it admits the representation

$$v_i = \int_a^b u_i(t) d\mu(t), \quad 1 \le i \le n\,.$$

Lemma 6.2 *A data sequence $(\boldsymbol{x}, \boldsymbol{y})$ is k-monotone if and only if the sequence of divided differences $\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y})$ is a moment sequence with respect to $\mathfrak{M}(\boldsymbol{x})$, the sequence of B-splines.*

Proof. By Lemma 3.1(3), $f \in \mathcal{M}_*^k(a, b)$ can be represented as

$$f(x) = p(x) + \frac{1}{k!} \int_a^b k(x - t)_+^{k-1} d\mu(t) \tag{6.1}$$

where $p \in \Pi_k$ and μ is a bounded non-decreasing function. If $f\big|_{\boldsymbol{x}} = \boldsymbol{y}$, then

$$v_i := [x_i, \ldots, x_{i+k}]\boldsymbol{y} = [x_i, \ldots, x_{i+k}]f = \frac{1}{k!} \int_a^b M_i(t) d\mu(t),$$

i.e., $\boldsymbol{v}$ is a moment sequence w.r.t. $\mathfrak{M}$.

Conversely, if for the sequences $\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y})$ and $\mathfrak{M}(\boldsymbol{x})$ there exists a bounded non-decreasing function μ such that

$$v_i = \frac{1}{k!} \int_a^b M_i(t)\, d\mu(t)\,, \quad i = 1, \ldots, n\,,$$

then, for any $p \in \Pi_k$, the function f defined by (6.1) is in $\mathcal{M}_*^k$ and satisfies

$$[x_i, \ldots, x_{i+k}]f = v_i := [x_i, \ldots, x_{i+k}]\boldsymbol{y}\,, \quad i = 1, \ldots, n\,. \tag{6.2}$$

Finally, in (6.1), we can choose $p \in \Pi_k$ so that the equality in (5.1) holds for $j = 1, \ldots, k$, and that together with (6.2) implies successively that it is also true for $j = k + 1, \ldots, n + k$, hence the sequence (x_i, y_i) is k-monotone. ∎

Now, we need a result from the theory of moments which gives a characterization of the moment sequences.

Definition 6.3 A sequence $\boldsymbol{v} \in \mathbb{R}^n$ of real numbers is called positive w.r.t. $\mathcal{U} = (u_i)_{i=1}^n$ (recall that $\mathcal{U}$ is a sequence of continuous linearly independent real valued functions on $[a, b]$) if

$$\sum_{i=1}^n a_i u_i(t) \ge 0\,, \quad a \le t \le b\,, \qquad \Rightarrow \qquad \sum_{i=1}^n a_i v_i \ge 0\,.$$

Theorem 6.4 (Krein & Nudelman [7, Theorem 3.1.1, p. 58]) *Let $\mathcal{U} := (u_i)_{i=1}^n$ be a sequence of continuous linearly independent real-valued functions on $I = [a, b]$ with the property that there exists a strictly positive polynomial $p \in \operatorname{span}\mathcal{U}$. A sequence $\boldsymbol{v} \in \mathbb{R}^n$ is a moment sequence w.r.t. $\mathcal{U}$ if and only if $\boldsymbol{v}$ is positive w.r.t. $\mathcal{U}$.*

Since $\operatorname{span}\mathfrak{M}(\boldsymbol{x})$ contains constants, we may combine this theorem with Lemma 6.2 to obtain the following criterion for k-monotonicity of data.

Corollary 6.5 *A data sequence $(\boldsymbol{x}, \boldsymbol{y})$ is k-monotone if and only if the sequence of divided differences $\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y})$ is positive w.r.t. $\mathfrak{M}(\boldsymbol{x})$,* i.e., *if and only if*

$$\sum_{i=1}^n a_i M_i(t) \ge 0 \quad \Rightarrow \quad \sum_{i=1}^n a_i v_i \ge 0\,, \quad v_i = [x_i, \ldots, x_{i+k}]\boldsymbol{y}\,.$$

7 Blending of k-monotone functions

In this section, we will give a partial solution to Problem 5.1. Namely, in Proposition 7.3, we prove that, provided f and g have sufficiently many points of intersection, a function $h \in \mathcal{M}^k[f,g]$ exists.

We need two auxiliary statements.

The following lemma is a particular case of Lemma 3.2 in Beatson [1] concerning the spline blending. Actually, we will use a more detailed statement which is formulated within the proof of Proposition 7.3.

Lemma 7.1 (Beatson [1]) *Let $k \in \mathbb{N}$, $n = 2k^2$ and let $p \in \Pi_k$ be a nonnegative polynomial on $[a,b]$. Then, for any knot sequence*

$$\mathbf{t} := \{a = t_0 \le t_1 \le \ldots \le t_n < t_{n+1} = b\}$$

there exists a nonnegative spline $s_2 \in \mathcal{S}_{\mathbf{t},k}(\mathbb{R})$ (i.e., *s_2 is a spline of order r on the knot sequence $\mathbf{t}$*) *such that*

$$s_2 \equiv 0 \quad \text{on} \quad (-\infty, a], \qquad 0 \le s_2 \le p \quad \text{on} \quad [a,b], \qquad s_2 = p \quad \text{on} \quad [b, \infty).$$

The next statement is a well-known property of divided differences.

Lemma 7.2 *Let $(x_j)_{j=1}^{n+k}$ be any non-decreasing sequence such that $x_j < x_{j+k}$. Then, for any subsequence $(x_{i_0}, \ldots, x_{i_k})$ of length $k+1$, there exist coefficients ν_j such that, for any continuous f (which is differentiable at the repeated knots),*

$$[x_{i_0}, \ldots, x_{i_k}]f = \sum_{j=1}^{n} \nu_j [x_j, \ldots, x_{j+k}]f.$$

Proposition 7.3 *For $k \in \mathbb{N}$ and $n = 2k^2$, let $f, g \in \mathcal{M}_*^k(a,b)$ be such that*

$$f(t_j) = g(t_j) \quad \textit{on} \quad \{a = t_0 < t_1 < \ldots < t_n < t_{n+1} = b\}$$

Then there exists a function $h \in \mathcal{M}_*^k(a,b)$ such that

$$h^{(l)}(a+) = f^{(l)}(a+), \quad h^{(l)}(b-) = g^{(l)}(b-), \quad l = 0, \ldots, k-1.$$

Note that the condition that all points t_i in the statement of Proposition 7.3 are distinct is not essential and is only used here in order to simplify the exposition.

Proof. Let us introduce two sequences $\boldsymbol{x} = (x_i)_{i=1}^{n+2k}$ and $\boldsymbol{y} = (y_i)_{i=1}^{n+2k}$:

$$x_j := \begin{cases} a\,, & 1 \le j \le k, \\ t_{j-k}\,, & k+1 \le j \le n+k, \\ b\,, & n+k+1 \le j \le n+2k; \end{cases} \qquad y_j := \begin{cases} f^{(j-1)}(a+)\,, & 1 \le j \le k, \\ f(x_j) = g(x_j)\,, & k+1 \le j \le n+k, \\ g^{(j-n-k-1)}(b-)\,, & n+k+1 \le j \le n+2k\,. \end{cases}$$

It is convenient to arrange this data set $(\boldsymbol{x}, \boldsymbol{y})$ as follows:

$$\begin{array}{ccccccccccc}
\boldsymbol{y}_1 & & & & & & & & & & \\
\shortparallel & & & & & & & & & & \\
(y_j)_{j=1}^{n+k+1} \rightarrow & f(a) & \ldots\; f^{(k-1)}(a) & f(x_{k+1}) & \ldots & f(x_{n+k}) & f(b) & & & & \\
 & \uparrow & \uparrow & \uparrow & & \uparrow & \uparrow & & & & \\
 & x_1 = & \ldots = x_k = a < & x_{k+1} & \le \ldots \le & x_{n+k} < & b = x_{n+k+1} & = \ldots = & x_{n+2k} & & \\
 & & \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow & & \\
 & & g(a) & g(x_{k+1}) & \ldots & g(x_{n+k}) & g(b) & \ldots & g^{(k-1)}(b) & \leftarrow (y_j)_{j=k}^{n+2k} & \\
 & & & & & & & & & \shortparallel & \\
 & & & & & & & & & \boldsymbol{y}_2 &
\end{array}$$

Set

$$\begin{aligned}
(7.1) \qquad \boldsymbol{x}_* &:= (x_1, \ldots, x_k, x_{n+k+1}, \ldots, x_{n+2k}) &&:= (\overbrace{a, \ldots, a}^{k}, \overbrace{b, \ldots, b}^{k})\,, \\
\boldsymbol{y}_* &:= (y_1, \ldots, y_k, y_{n+k+1}, \ldots, y_{n+2k}) &&:= \big(f(a), \ldots, f^{(k-1)}(a), g(b), \ldots, g^{(k-1)}(b)\big)\,.
\end{aligned}$$

We need to interpolate $\boldsymbol{y}_*$ on $\boldsymbol{x}_*$ by a k-monotone function h. Denote by

$$\mathfrak{M}(\boldsymbol{x}_*) =: (B_i)_{i=1}^k, \quad \boldsymbol{v}(\boldsymbol{x}_*, \boldsymbol{y}_*) =: (w_i)_{i=1}^k$$

the sequences of the B-splines and of divided differences, respectively, which correspond to $(\boldsymbol{x}_*, \boldsymbol{y}_*)$. By Corollary 6.5, existence of a k-monotone interpolant h to the data (7.1) will follow if we show that

$$\sum_{i=1}^k a_i B_i(t) \ge 0 \quad \Rightarrow \quad \sum_{i=1}^k a_i w_i \ge 0. \tag{7.2}$$

We start with some preliminaries.

1) Let $(v_j)_{j=1}^{n+k}$ and $(M_j)_{j=1}^{n+k}$ be the sequences of divided differences and B-splines, respectively, constructed with respect to the entire set $(x_j, y_j)_{j=1}^{n+2k}$. Consider two sets of the following subsequences:

$$\begin{aligned} &\boldsymbol{x}_1 := (x_j)_{j=1}^{n+k+1}, \quad \boldsymbol{y}_1 := (y_j)_{j=1}^{n+k+1}, \quad \boldsymbol{v}_1 := (v_j)_{j=1}^{n+1}, \quad \mathfrak{M}_1 := (M_j)_{j=1}^{n+1}; \\ &\boldsymbol{x}_2 := (x_j)_{j=k}^{n+2k}, \quad\ \ \boldsymbol{y}_2 := (y_j)_{j=k}^{n+2k}, \quad\ \ \boldsymbol{v}_2 := (v_j)_{j=k}^{n+k}, \quad \mathfrak{M}_2 := (M_j)_{j=k}^{n+k}. \end{aligned} \tag{7.3}$$

By assumption, k-monotone f interpolates $\boldsymbol{y}_1$ on $\boldsymbol{x}_1$, and k-monotone g interpolates $\boldsymbol{y}_2$ on $\boldsymbol{x}_2$, thus, both sets of data $(\boldsymbol{x}_\nu, \boldsymbol{y}_\nu)$, $\nu = 1, 2$, are k-monotone. Then Corollary 6.5 implies that

$$\boldsymbol{v}_\nu \text{ is positive w.r.t. } \mathfrak{M}_\nu, \quad \nu = 1, 2. \tag{7.4}$$

2) Since $(v_j), (M_j)$ are divided differences of certain functions on $\boldsymbol{x}$, while $(w_i), (B_i)$ are divided differences of the same functions on $\boldsymbol{x}_* \subset \boldsymbol{x}$, by Lemma 7.2, there exist expansions

$$w_i = \sum_{j=1}^{n+k} c_{ij} v_j, \quad B_i(x) = \sum_{j=1}^{n+k} c_{ij} M_j(x)$$

with the same coefficients (c_{ij}) in both of these equations. This implies that, for any $(a_i)_{i=1}^k \subset \mathbb{R}$, the expansions

$$\sum_{i=1}^k a_i B_i(x) = \sum_{j=1}^{n+k} c_j M_j(x), \quad \sum_{i=1}^k a_i w_i = \sum_{j=1}^{n+k} c_j v_j,$$

have the same coefficients $c_j = \sum_{i=1}^k a_i c_{ij}$.

3) The B-splines $(B_i) \in \mathfrak{M}(\boldsymbol{x}_*)$ have the form

$$B_i(t) := k\,[\overbrace{a, \dots, a}^{k+1-i}, \overbrace{b, \dots, b}^{i}](\cdot - t)_+^{k-1} = \frac{k}{(b-a)^k} \binom{k-1}{i-1} (t-a)^{i-1} (b-t)^{k-i}, \quad i = 1, \dots, k,$$

i.e., they are Bernstein basis polynomials of order k, so that

$$\sum a_i B_i \in \Pi_k, \quad \forall (a_i).$$

Now, let us prove (7.2). Suppose that, for some sequence (a_i),

$$p_a(x) := \sum_{i=1}^k a_i B_i(x) \ge 0.$$

Since p_a is a polynomial of order k, and

$$p_a(x) := \sum_{i=1}^k a_i B_i(x) = \sum_{j=1}^{n+k} c_j M_j(x) \ge 0, \quad \text{and} \quad n \ge 2k^2,$$

the method of the proof of Beatson's Smoothing Lemma [1, Lemma 3.2] shows that there is an index l, $k \le l \le n+1$, such that

$$s_1(x) := \sum_{j=1}^{l} c_j M_j(x) \ge 0, \quad s_2(x) := \sum_{j=l+1}^{n+k} c_j M_j(x) \ge 0. \tag{7.5}$$

(We will not repeat Beatson's argument here and only mention that the sign variation diminishing property of B-spline series (see [4, Section 5.10], for example) as well as their finite support are used.) ¿From definitions (7.3) it follows that $s_\nu \in \mathfrak{M}_\nu$, $\nu = 1, 2$, which allows us to conclude that, since $\boldsymbol{v}_\nu$ are positive w.r.t. $\mathfrak{M}_\nu$ (see (7.4)), (7.5) implies

$$\sum_{j=1}^{l} c_j v_j \ge 0, \quad \sum_{j=l+1}^{n+k} c_j v_j \ge 0.$$

Finally,

$$\sum_{i=1}^{k} a_i w_i = \sum_{j=1}^{n+k} c_j v_j = \sum_{i=1}^{l} c_j v_j + \sum_{j=l+1}^{n+k} c_j v_j \ge 0.$$

Hence, (7.2) is proved, and the proof of the proposition is now complete. ∎

Now, having proved existence of a function $h \in \mathcal{M}^k[f, g]$, we may use Lemma 3.4 to derive existence of a spline $z \in \mathcal{M}^k[f, g]$.

Corollary 7.4 *For $k \in \mathbb{N}$, $n = 2k^2$, let $f, g \in \mathcal{M}_*^k(a, b)$ be such that*

$$f(t_j) = g(t_j) \quad \text{on} \quad \{a = t_0 < t_1 < \ldots < t_n < t_{n+1} = b\}.$$

Then, there exists a spline z such that

$$z \in \mathcal{S}_{k',k} \cap \mathcal{M}^k[f, g].$$

Note that, for $k = 1$ or 2, that is for monotone or convex functions f and g, a procedure of k-monotone blending of f and g is quite evident geometrically.

8 Auxiliary Whitney type estimates

In this section, we give some Whitney type estimates for approximation of polynomials $\mathfrak{p} \in \Pi_r$ by splines and polynomials of degree k.

As usual, $\omega_k(f, \delta, I)_p$ denotes the kth modulus of smoothness of f with the step δ on the interval I,

$$\omega_k(f, \delta, I)_p := \sup_{0<h\le\delta} \left\| \Delta_h^k(f, \cdot, I) \right\|_{\mathbb{L}_p(I)},$$

where $\Delta_h^k(f, x, I)$ is the kth forward difference,

$$\Delta_h^k(f, x, I) := \begin{cases} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x + ih), & \text{if } [x, x+kh] \subset I, \\ 0, & \text{otherwise.} \end{cases}$$

It is also convenient to denote

$$\omega_k(f)_p := \omega_k(f)_{\mathbb{L}_p(I)} := \omega_k(f, |I|, I)_p.$$

Lemma 8.1 *Let $k, r \in \mathbb{N}$, $0 < p \le \infty$, $I = (a, b)$, $\mathfrak{p} \in \Pi_r$, and let s be a spline of order k with at most $C(k)$ pieces in I* (i.e., $s \in \mathcal{S}_{C(k),k}$). *Then,*

$$\|\mathfrak{p} - s\|_p \ge c_{p,r,k}\, \omega_k(\mathfrak{p})_p.$$

Proof. Let J be a largest subinterval of I between two successive knots of s (and, hence, $|J|/|I| \geq 1/C(k)$), and let $\mathfrak{q} \in \Pi_k$ be the restriction of s to J. Then, using Whitney's inequality,

$$E(\mathfrak{p}, \Pi_k)_{\mathbb{L}_p(I)} \overset{p,k}{\sim} \omega_k(\mathfrak{p})_{\mathbb{L}_p(I)}, \tag{8.1}$$

and the Markov's type inequality (see [4, (4.2.10) and (4.2.16)])

$$\|\mathfrak{p}\|_{\mathbb{L}_p(I)} \leq c_{p,r}(|I|/|J|)^{r-1+1/p}\|\mathfrak{p}\|_{\mathbb{L}_p(J)}\,,$$

we find

$$\begin{aligned} \|\mathfrak{p}-s\|_{\mathbb{L}_p(I)} &\geq \|\mathfrak{p}-\mathfrak{q}\|_{\mathbb{L}_p(J)} \geq c_{p,r}(|J|/|I|)^{r-1+1/p}\|\mathfrak{p}-\mathfrak{q}\|_{\mathbb{L}_p(I)} \\ &\geq c_{p,r,k}E(\mathfrak{p},\Pi_k)_{\mathbb{L}_p(I)} \geq c'_{p,r,k}\omega_k(\mathfrak{p})_p\,. \end{aligned}$$

■

Lemma 8.2 *Let $k, r \in \mathbb{N}$, $0 < p \leq \infty$, $I = (a, b)$, $\mathfrak{p} \in \Pi_r$, and let $l_k(\mathfrak{p})$ be the Lagrange polynomial of order k interpolating $\mathfrak{p}$ at any k (not necessarily distinct) points inside I. Then,*

$$\|\mathfrak{p} - l_k(\mathfrak{p})\|_p \leq c_{p,r,k}\omega_k(\mathfrak{p})_p\,.$$

Proof. Taking into account Lebesgue's inequality

$$\|\mathfrak{p} - l_k(\mathfrak{p})\|_p \leq \left(\sup_{\mathfrak{q}\in\Pi_r} \frac{\|l_k(\mathfrak{q})\|_p}{\|\mathfrak{q}\|_p} + 1\right) E(\mathfrak{p}, \Pi_k)_p$$

and Whitney's inequality (8.1), it suffices to prove that

$$\|l_k(\mathfrak{q})\|_p \leq c_{p,r,k}\,\|\mathfrak{q}\|_p, \quad \forall \mathfrak{q} \in \Pi_r.$$

We make use of Markov's inequality

$$\|\mathfrak{q}^{(k)}\|_\infty \leq c_{p,r,k}|I|^{-k-1/p}\|\mathfrak{q}\|_p$$

and the well–known error bound for the Lagrange interpolation

$$\|f - l_k(f)\|_\infty \leq c_k|I|^k\|f^{(k)}\|_\infty$$

to obtain

$$\|\mathfrak{q} - l_k(\mathfrak{q})\|_p \leq |I|^{1/p}\|\mathfrak{q} - l_k(\mathfrak{q})\|_\infty \leq c_k|I|^{1/p}|I|^k\|\mathfrak{q}^{(k)}\|_\infty \leq c_{p,r,k}\|\mathfrak{q}\|_p\,.$$

■

Lemma 8.3 *Let $k \in \mathbb{N}$, $f \in \mathcal{M}^k(a, b)$, and let $l_k(f, x; x_1, \ldots, x_k)$ be the Lagrange (Hermite-Taylor) polynomial of degree $\leq k-1$ interpolating f at the points x_i, $1 \leq i \leq k$, where $a =: x_0 < x_1 \leq \ldots \leq x_k < x_{k+1} := b$. Then*

$$(-1)^{k-i}\,(f(x) - l_k(f, x; x_1, \ldots, x_k)) \geq 0\,, \quad x \in (x_i, x_{i+1})\,, \quad i = 0, \ldots, k\,.$$

In other words, $f - l_k$ changes sign at $x_1, \ldots, x_k$.

Proof. First of all, if all the points x_i, $1 \leq i \leq k$, are distinct, this is Theorem 5 in Bullen [3].

In the case when some of x_i (but not all) coincide, the statement of the lemma is a consequence of the following result which follows from [4, Theorem 4.6.3]: *For a given $f \in \mathbb{C}^{(k-2)}(a, b)$, the Lagrange-Hermite polynomial $l_k(X) = l_k(f, x; x_1, \ldots, x_k)$ is a continuous function of $X = (x_1, \ldots, x_k)$ at each point $X^* = (x_1^*, \ldots, x_k^*) \in (a, b)^k$ such that not all x_i^*, $i = 1, \ldots, k$, are the same.*

In the case, when all points coincide, *i.e.*, , $x_1 = \ldots = x_k = \xi$, lemma follows from the following statement which can be proved by induction on k: *Let $k \in \mathbb{N}$, $f \in \mathcal{M}^k(a,b)$ and $\xi \in (a,b)$. If t_k is a Taylor polynomial of degree $\le k-1$ for f at ξ,* i.e., , *$t_k^{(i)}(\xi) = f^{(i)}(\xi\pm)$ for $i = 0,\ldots,k-1$ (or, more precisely, $t_k^{(i)}(\xi) = f^{(i)}(\xi)$ for $i = 0,\ldots,k-2$ and $t_k^{(k-1)}(\xi)$ is either $f^{(k-1)}(\xi+)$ or $f^{(k-1)}(\xi-)$), then,*

$$f(x) - t_k(x) \ge 0,\ x \in (\xi, b)\,, \quad \text{and} \quad (-1)^k\,(f(x) - p_k(x)) \ge 0,\ x \in (a,\xi)\,.$$

■

The following is an immediate corollary of Lemma 8.3.

Corollary 8.4 *For $k \in \mathbb{N}$, $f \in \mathcal{M}^k_*(a,b)$, and for a set of interpolation points $\{a = x_0 \le \ldots \le x_k = b\}$, let*

$$l_k := l_k(f; x_0 \ldots, x_{k-1}) \quad \text{and} \quad \widetilde{l}_k := \widetilde{l}_k(f, x_1 \ldots, x_k)$$

be two Lagrange (Hermite-Taylor) interpolants to f on the given sets. Then f lies between l_k and $\widetilde{l}_k$ on $[a,b]$, i.e.,

$$\min\{l_k, \widetilde{l}_k\} \le f \le \max\{l_k, \widetilde{l}_k\}.$$

Lemma 8.5 *Let $k, r \in \mathbb{N}$, $0 < p \le \infty$, $I = (a,b)$, $\mathfrak{p} \in \Pi_r \cap \mathcal{M}^k$, $0 \le \mu \le k-1$, and let $g \in \mathcal{M}^k$ be a function such that*

(8.2)
$$g^{(i)}(a) = \mathfrak{p}^{(i)}(a)\,, \quad i = 0,\ldots,\mu$$

and

(8.3)
$$g^{(i)}(b) = \mathfrak{p}^{(i)}(b)\,,\ i = 0,\ldots,k-\mu-1\,.$$

(Here, in the cases $\mu = 0$ and $\mu = k-1$, $g^{(k-1)}(b)$ and $g^{(k-1)}(a)$ are understood as $g^{(k-1)}(b-)$ and $g^{(k-1)}(a+)$, respectively.) Then,

$$\|\mathfrak{p} - g\|_p \le c_{p,r,k}\omega_k(\mathfrak{p})_p\,.$$

Proof. Consider the following Lagrange (Hermite–Taylor) polynomials of order k on $[a,b]$:

$$l_k := l_k(\mathfrak{p}; \overbrace{a,\ldots,a}^{\mu+1}, \overbrace{b,\ldots,b}^{k-\mu-1}) \quad \text{and} \quad \widetilde{l}_k := \widetilde{l}_k(\mathfrak{p}; \overbrace{a,\ldots,a}^{\mu}, \overbrace{b,\ldots,b}^{k-\mu})\,.$$

By Corollary 8.4, both k-monotone functions $\mathfrak{p}$ and g lie between l_k and $\widetilde{l}_k$ in $[a,b]$, *i.e.*,

$$\min\{l_k, \widetilde{l}_k\} \le \min\{\mathfrak{p}, g\} \le \max\{\mathfrak{p}, g\} \le \max\{l_k, \widetilde{l}_k\}\,.$$

Therefore,

$$\|g - \mathfrak{p}\|_p \le \|l_k - \widetilde{l}_k\|_p \le c_p\|l_k - \mathfrak{p}\|_p + c_p\|\mathfrak{p} - \widetilde{l}_k\|_p \le c_{p,r,k}\omega_k(\mathfrak{p})_p\,,$$

where the last inequality follows from Lemma 8.2. ■

In our proof, we need a slightly stronger statement in the case $\mu = 0$.

Lemma 8.6 *Let $k, r \in \mathbb{N}$, $0 < p \le \infty$, $I = (a,b)$, $\mathfrak{p} \in \Pi_r \cap \mathcal{M}^k$, and let $h \in \mathcal{M}^k$ be a function such that*

(8.4)
$$h(a) = \mathfrak{p}(a) \quad \text{and} \quad h^{(i)}(b) = \mathfrak{p}^{(i)}(b)\,,\ i = 0,\ldots,k-2\,, \quad h^{(k-1)}(b-) \le \mathfrak{p}^{(k-1)}(b)\,.$$

Then

(8.5)
$$\|\mathfrak{p} - h\|_p \le c_{p,r,k}\omega_k(\mathfrak{p})_p\,.$$

Proof. First of all, assume that there exists $\delta > 0$ such that $\mathfrak{p} \in \mathcal{M}^k(a, b+\delta)$, and set

$$g = \begin{cases} h, & \text{on} \quad [a, b) \,, \\ \mathfrak{p}, & \text{on} \quad [b, b+\delta] \,. \end{cases}$$

Then g is k-monotone on $[a, b+\delta]$ and satisfies all other assumptions of Lemma 8.5 (with $\mu = 0$), hence

$$\|g - \mathfrak{p}\|_{\mathbb{L}_p[a,b+\delta]} \le c_{p,r,k}\omega_k(\mathfrak{p})_{\mathbb{L}_p[a,b+\delta]} \,.$$

Letting $\delta \to 0$, we obtain

$$\|h - \mathfrak{p}\|_{\mathbb{L}_p[a,b]} \le \lim_{\delta\to 0} \|g - \mathfrak{p}\|_{\mathbb{L}_p[a,b+\delta]} \le c_{p,r,k} \lim_{\delta\to 0} \omega_k(\mathfrak{p})_{\mathbb{L}_p[a,b+\delta]} = c_{p,r,k}\, \omega_k(\mathfrak{p})_{\mathbb{L}_p[a,b]} \,.$$

Now, if for any $\delta > 0$, $\mathfrak{p} \notin \mathcal{M}^k(a, b+\delta)$ one can replace $\mathfrak{p}$ by

$$\widetilde{\mathfrak{p}}(x) := \mathfrak{p}(x) + \epsilon(x-a)(x-b)^{k-1} \,.$$

Then, $\widetilde{\mathfrak{p}} \in \Pi_{\max\{r,k+1\}} \cap \mathcal{M}^k(a, b+\Delta)$ for some $\Delta > 0$,

$$\widetilde{\mathfrak{p}}(a) = \mathfrak{p}(a) \,, \quad \widetilde{\mathfrak{p}}^{(i)}(b) = \mathfrak{p}^{(i)}(b), \ 0 \le i \le k-2,$$

and

$$\widetilde{\mathfrak{p}}^{(k-1)}(b) = \mathfrak{p}^{(k-1)}(b) + (k-1)!\epsilon(b-a) \ge \mathfrak{p}^{(k-1)}(b) \ge h^{(k-1)}(b-) \,.$$

Now, using the same argument as above and letting $\epsilon \to 0$ and $\Delta \to 0$ completes the proof of the lemma. ■

9 Proof of Proposition 4.2

The following statement summarizes results of §§5-8.

Proposition 9.1 *Let $k \in \mathbb{N}$, $n = 2k^2$, $0 < p \le \infty$, $I = (a, b)$, $\mathfrak{p} \in \Pi_r \cap \mathcal{M}^k$, and let $g_* \in \mathcal{S}_{C(k),k} \cap \mathcal{M}^k_*$ be such that*

$$g_*(t_j) = \mathfrak{p}(t_j) \quad on \quad \{a = t_0 < t_1 < \ldots < t_n < t_{n+1} = b\}$$

Then, there exists a spline z such that

$$z \in \mathcal{S}_{k',k} \cap \mathcal{M}^k[g_*, \mathfrak{p}]$$

and

$$\|\mathfrak{p} - z\|_p \le c_2 \|\mathfrak{p} - g_*\|_p, \quad c_2 = c_2(p, r, k) \,. \tag{9.1}$$

Proof. First of all, Corollary 7.4 implies that there exists a spline $z \in \mathcal{S}_{k',k} \cap \mathcal{M}^k[g_*, \mathfrak{p}]$. Now, since z satisfies condition (8.4) of Lemma 8.6 (which follows from the definition of the class $\mathcal{M}^k[g_*, \mathfrak{p}]$ and the fact that $g_*(a) = \mathfrak{p}(a)$) we have the estimate

$$\|\mathfrak{p} - z\|_p \le c_{p,r,k}\omega_k(\mathfrak{p})_p \,.$$

On the other hand, for $g_* \in \mathcal{S}_{C(k),k}$, Lemma 8.1 yields

$$\|\mathfrak{p} - g_*\|_p \ge c_{p,r,k}\omega_k(\mathfrak{p})_p \,.$$

Combining both estimates we obtain (9.1). ■

Remark 9.2 Applying Proposition 9.1 to $\widetilde{\mathfrak{p}}(t) := (-1)^k \mathfrak{p}(-t)$ and $\widetilde{g}_*(t) := (-1)^k g_*(-t)$ we conclude that there also exists a spline $\widetilde{z} \in \mathcal{S}_{k',k} \cap \mathcal{M}^k[\mathfrak{p}, g_*]$ for which (9.1) is valid.

Also, we need the following elementary statement.

Lemma 9.3 *Let $(x_j)_{j=1}^{\infty}$ be such that $x_i \neq x_j$ if $i \neq j$, and $\lim_{j\to\infty} x_j = L$, and let, for some $k \geq 2$, f be $(k-2)$-times continuously differentiable in some ϵ-neighborhood of L and have onesided $(k-1)$st derivatives at L. If $f(x_j) = 0$ for all j, then $f^{(i)}(L) = 0$ for $i = 0, \dots, k-2$ and either $f^{(k-1)}(L+) = 0$ or $f^{(k-1)}(L-) = 0$.*

Proof of Proposition 4.2. If $0 < p < \infty$, let $\mathfrak{f}_*$ be a best $\mathbb{L}_p$-approximant to $\mathfrak{p} \in \Pi_r \cap \mathcal{M}^k$ from the set $\mathcal{M}^k[f]$ whose existence is guaranteed by Lemma 3.3, and so Lemma 3.8 is valid. If $p = \infty$, we choose $\mathfrak{f}_*$ to be a best $\mathbb{L}_\infty$-approximant to $\mathfrak{p}$ from the set $\mathcal{M}^k[f]$ which satisfies Lemma 3.9.

We need to prove that there exists a spline s such that

$$s \in \mathcal{S}_{C(k),r} \cap \mathcal{M}^k[f] \tag{9.2}$$

and

$$\|\mathfrak{p} - s\|_p \leq c_2 \|\mathfrak{p} - \mathfrak{f}_*\|_p. \tag{9.3}$$

Lemmas 3.8 and 3.9 imply that, on any interval (c, d) where the difference $\mathfrak{f}_*(x) - \mathfrak{p}(x)$ has exactly $m-1$ distinct zeros, we have

$$\mathfrak{f}_* \in \mathcal{S}_{mk',k}, \quad k' = \lfloor k/2 \rfloor + 1\,. \tag{9.4}$$

Denote by $\mathfrak{Z}$ the set of all zeros of the function $\mathfrak{f}_* - \mathfrak{p}$, *i.e.,*

$$\mathfrak{Z} := \left\{ x \in [a,b] \middle| \, \mathfrak{f}_*(x) = \mathfrak{p}(x) \right\}\,,$$

and let $\mathfrak{Z}^*$ be the set of all limit points of $\mathfrak{Z}$. Also, let $\#\mathfrak{Z}$ denote the cardinality of $\mathfrak{Z}$. (Note, that the set $\mathfrak{Z}$ does not take into account multiplicity of zeros. This is not essential, and is only done to simplify the exposition.)

The proof is quite transparent. If $\mathfrak{Z}$ consists of only a few (less than $4k^2+4$) points, (9.4) implies that $\mathfrak{f}_*$ has to be in $\mathcal{S}_{C(k),k}$, and so there is nothing to prove. If $\#\mathfrak{Z}$ is not less than $4k^2+4$ but is finite, we use Proposition 9.1 to blend $\mathfrak{f}_*$ and $\mathfrak{p}$ on intervals containing the first and the last $2k^2+2$ points from $\mathfrak{Z}$ (and, hence, $\mathfrak{f}_*$ which has many "knots" between these intervals is replaced by the polynomial $\mathfrak{p}$ there). Finally, if $\mathfrak{Z}$ is an infinite set, the set $\mathfrak{Z}^*$ is necessarily not empty and connected. Hence, $\mathfrak{Z}^*$ is a closed subinterval of (or a point in) $[a,b]$. We will show that $\mathfrak{f}_* \equiv \mathfrak{p}$ on $\mathfrak{Z}^*$, and so it'll remain to apply the above mentioned argument in the case $\#\mathfrak{Z} < \infty$ to the set $[a,b] \setminus \mathfrak{Z}^*$ which is a union of at most two intervals.

We now fill in the details, and consider the following three cases.

Case 1: $\#\mathfrak{Z} < 4k^2 + 4$.

According to (9.4),

$$\mathfrak{f}_* \in \mathcal{S}_{C(k),k}, \quad C(k) \leq (4k^2+4)k',$$

so we let $s = \mathfrak{f}_*$.

Case 2: $4k^2 + 4 \leq \#\mathfrak{Z} < \infty$

Denote by $I_\nu := [a_\nu, b_\nu]$, $\nu = 1, 2$, the smallest closed subintervals of $[a,b]$ which contain the first and the last $2k^2+2$ points of $\mathfrak{Z}$, respectively (*i.e.,* $a_1 = \min(\mathfrak{Z})$ and $b_2 = \max(\mathfrak{Z})$). By (9.4), $\mathfrak{f}_* \in \mathcal{S}_{(2k^2+1)k',k}(I_\nu)$, $\nu = 1, 2$, and, hence, by Proposition 9.1 and Remark after it we conclude that there exists two splines s_1, s_2 such that

$$s_1 \in S_{k',k} \cap \mathcal{M}^k[\mathfrak{f}_*, \mathfrak{p}](I_1), \quad s_2 \in S_{k',k} \cap \mathcal{M}^k[\mathfrak{p}, \mathfrak{f}_*](I_2),$$

and

$$\|\mathfrak{p} - s_\nu\|_{\mathbb{L}_p(I_\nu)} \leq c_2 \|\mathfrak{p} - \mathfrak{f}_*\|_{\mathbb{L}_p(I_\nu)}\,. \tag{9.5}$$

Also, note that $\mathfrak{f}_* \in \mathcal{S}_{k',k}[a, a_1]$ and $\mathfrak{f}_* \in \mathcal{S}_{k',k}[b_2, b]$, and define

$$s(x) = \begin{cases} \mathfrak{f}_*(x)\,, & x \in [a, a_1] \cup [b_2, b]\,, \\ s_1(x)\,, & x \in [a_1, b_1]\,, \\ \mathfrak{p}(x)\,, & x \in [b_1, a_2]\,, \\ s_2(x)\,, & x \in [a_2, b_2]\,. \end{cases}$$

Then,

$$s \in \mathcal{S}_{C(k),r} \cap \mathcal{M}^k[f](a, b), \quad C(k) \le 4k' + 1,$$

and, clearly, (9.3) is satisfied.

Case 3: $\#\mathfrak{Z} = \infty$

Clearly, the set of all limit points $\mathfrak{Z}^*$ is not empty in this case. Also, $\mathfrak{Z}^*$ is closed, and we now show that it has to be connected. This will imply that $\mathfrak{Z}^* = [c, d] \subset [a, b]$ (not excluding the possibility that $c = d$). Taking into account that $\mathfrak{f}_* - \mathfrak{p}$ is $(k-2)$-times continuously differentiable and has onesided $(k-1)$st derivatives on $[a, b]$ (which is guaranteed by the assumption that $\mathfrak{f}_* \in \mathcal{M}^k[f]$), we apply Lemma 9.3 to conclude that, for every $x \in \mathfrak{Z}^*$, at least one of two relations takes place:

$$\mathfrak{f}_*^{(i)}(x\pm) = \mathfrak{p}^{(i)}(x)\,, \quad i = 0, \ldots, k-1\,.$$

Thus, if $\{c, d\} \subset \mathfrak{Z}^*$, then $\mathfrak{p} \in \mathcal{M}^k[\mathfrak{f}_*](c, d)$, so that the function

$$g_*(x) = \begin{cases} \mathfrak{f}_*(x) & x \in [a, b] \setminus [c, d]\,, \\ \mathfrak{p}(x)\,, & x \in [c, d] \end{cases}$$

is in $\mathcal{M}^k[\mathfrak{f}_*](a, b) \subset \mathcal{M}^k[f](a, b)$. Also, if $\mathfrak{f}_* \not\equiv \mathfrak{p}$ on $[c, d]$, then g_* approximates $\mathfrak{p}$ better (in the $\mathbb{L}_p$-metric) than $\mathfrak{f}_*$ on $[a, b]$ if $0 < p < \infty$ and not worse than $\mathfrak{f}_*$ if $p = \infty$. Therefore, we know (can assume) that $\mathfrak{f}_* \equiv \mathfrak{p}$ on $[c, d]$, hence $[c, d] \subset \mathfrak{Z}^*$.

Thus, we can assume that $\mathfrak{Z}^* = [c, d]$ for some $[c, d] \subset [a, b]$. We also assume that $a < c \le d < b$, the cases when $c = a$ or $d = b$ being analogous (and simpler).

Since $(a, c) \cap \mathfrak{Z}^* = \emptyset$, any closed subinterval of (a, c) contains finitely many points from $\mathfrak{Z}$.

Now, if $\#\left((a, c) \cap \mathfrak{Z}\right) < 2k^2 + 2$, (9.4) implies that $\mathfrak{f}_* \in \mathcal{S}_{(2k^2+2)k',k}[a, c]$ and we define the spline s_1 to be $\mathfrak{f}_*$ on $[a, c]$.

If, on the other hand, $\#\left((a, c) \cap \mathfrak{Z}\right) \ge 2k^2 + 2$, then there exists $c' \in (a, c)$ such that $c' \in \mathfrak{Z}$, and the interval (a, c') contains exactly $2k^2 + 1$ points from $\mathfrak{Z}$. The same construction as in Case 2 allows us to obtain a k-monotone spline $\tilde{s}_1 \in \mathcal{S}_{2k',k}(a, c') \cap \mathcal{M}^k[\mathfrak{f}_*, \mathfrak{p}]$ which "blends" $\mathfrak{f}_*$ with $\mathfrak{p}$ (in a k-monotone fashion) on (a, c'), and approximates $\mathfrak{p}$ as well as $\mathfrak{f}_*$. Now, we define s_1 by

$$s_1(x) = \begin{cases} \tilde{s}_1(x)\,, & x \in [a, c']\,, \\ \mathfrak{p}(x)\,, & x \in [c', c]\,. \end{cases}$$

The same argument can now be used "at the right end" to yield a construction of $s_2 \in \mathcal{S}_{(2k^2+2)k',k}[d, b]$ satisfying all conditions required.

Finally, we set

$$s(x) = \begin{cases} s_1(x)\,, & x \in [a, c]\,, \\ \mathfrak{p}(x)\,, & x \in [c, d]\,, \\ s_2(x)\,, & x \in [d, b]\,. \end{cases}$$

Then,

$$s \in \mathcal{S}_{C(k),k} \cap \mathcal{M}^k[f][a, b], \quad C(k) \le (4k^2 + 4)k' + 1\,,$$

which completes the proof of Case 3, and of Proposition 4.2. ∎

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