On L_∞-boundedness of the L_2-projector onto splines with multiple knots

A. Yu. Shadrin

Institut für Geometrie und Praktische Mathematik
RWTH, 52056 Aachen, Germany

Abstract

With the help of the MAPLE tools we prove that the L_∞-norm of the L_2-projector onto C^m-splines of order k is bounded independently of Δ for small m and $k \ge k_0(m)$, namely,

$$\sup_\Delta \|P_{S_{k,m}(\Delta)}\|_\infty < c_k, \qquad m = 1,\, k \ge 5; \quad m = 2,\, k \ge 10.$$

This gives new evidence to C.de Boor's conjecture which states such a boundedness for any $0 \le m \le k-2$, and which has been proved so far only for $m = 0, k \in \mathbf{N}$ and $m = 1, k = 3$.

We reduced the problem to computing the largest eigenvalue of certain $(2m+2)\times(2m+2)$ matrix, and used the MAPLE for these computations. This scheme is also applicable for estimating the norms of P_S for $m > 2$ and $k \ge (m+1)^2 + 1$. For example, the mesh-independent boundedness of $\|P_{S_{k,m}(\Delta)}\|_\infty$ holds also, if

$$m = 3,\, k = 17, 18; \qquad m = 4,\, k = 26, 27.$$

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1. Introduction

For any integer k, l, m with $l + m = k - 1$, and any partition of the interval $[a, b]$

$$\Delta = \{a = t_0 < t_1 < \ldots < t_N = b\}$$

denote by

$$S := S_k^l(\Delta) := S_{k,m}(\Delta)$$

the space of polynomial splines of order k (of degree $< k$) and smoothness m (with deficiency $l = k-1-m$). The latter means that $S_{k,-1}(\Delta) := P_k(\Delta)$ is the space of piecewise polynomial functions (of maximal deficiency k) and

$$S_{k,m}(\Delta) := P_k(\Delta) \cap C^m[a, b].$$

Traditional splines with highest smoothness at breakpoints belong to C^{k-2} and have the minimal deficiency 1, we denote their space by

$$S_k(\Delta) := S_k^1(\Delta) := S_{k,k-2}(\Delta).$$

Consider the operator P_S of the orthoprojection onto S defined by

$$\int_a^b [f(t) - P_S(f, t)]\sigma(t)\, dt = 0, \quad \forall \sigma \in S.$$

We are interested in P_S as an operator from L_p onto L_p, namely in bounds for its norm

$$\|P_S\|_p := \sup_{\|f\|_p = 1} \|P_S(f)\|_p, \quad 1 \le p \le \infty.$$

In 1972 C.de Boor [B2] conjectured that the L_∞-norms of these projectors with respect to splines of the minimal deficiency are bounded independently of Δ.

Conjecture. *For any* $k \in \mathbf{N}$

$$\sup_\Delta \|P_{S_k(\Delta)}\|_\infty < c_k. \tag{1.1}$$

The conjecture was based *only on two* cases treated by that time:

$k = 2$, Z. Ciesielsky [C], 1963

$k = 3$, C. de Boor [B1], 1968

Since then *none* of the other k's (with deficiency 1) has been added to the list.

However, studies of this problem gave a series of particular results which played an important role in the univariate spline theory as a whole. We will not discuss this connection in details, but concentrate on the problem itself.

Since

$$\sup_\Delta \|P_{S_k(\Delta)}\|_\infty = \sup_{\Delta, m, p} \|P_{S_{k,m}(\Delta)}\|_p,$$

a consequence of (1.1) is the estimate

$$\|P_{S_{k,m}(\Delta)}\|_p < c_k, \tag{1.2}$$

so one may test (1.1) testing (1.2) with various particular parameters k, m, p, Δ.

1.1. Mesh-dependent bounds. In the series of papers the L_∞-boundedness of P_S was established for the quasi-uniform and not much growing quasi-geometric meshes. Precisely, for

$$h_i := t_{i+1} - t_i, \quad \kappa_i := t_{i+k} - t_i,$$

set

$$\Delta_M := \{\Delta : \sup_{ij} \kappa_i/\kappa_j \le M\}, \qquad \Delta_\rho := \{\Delta : \sup_{|i-j|=1} h_i/h_j \le \rho\}.$$

Then

$$\sup_{\Delta\subset\Delta_M} \|P_{S_k(\Delta)}\|_\infty < c_k M^{1/2}; \qquad \sup_{\Delta\subset\Delta_\rho} \|P_S\|_\infty < c(k,\rho), \quad \rho < 1+\epsilon_k. \tag{1.3}$$

The basis for these estimates was the fact that the spline orthoprojection of function with finite support decays exponentially out of this support. This fact was discovered by Douglas, Dupont & Wahlbin [DDW], generalized by de Boor [B3], and found its mostly complete (and spline-free) form in Demko's theorem [De] for inverses of the band matrices.

However, (1.3) do not say anything *pro* conjecture, since such a mesh restriction provides L_∞-boundedness of similar (interpolation) spline pojectors which are known to be unbounded in general.

In particular, most of these projectors are unbounded on some classes of strictly geometric meshes

$$\Delta'_\rho = \{\Delta : h_{i+1}/h_i = \rho\}.$$

In this respect, *pro* conjecture were results of Höllig [H], Feng & Kozak [FK], and Jia [J1], that

$$\sup_{\rho;\,\Delta\subset\Delta'_\rho} \|P_{S_{k,m}(\Delta)}\|_\infty < c_k.$$

But their methods worked strictly for this geometric case.

There are yet two other approaches to be mentioned.

The first, by Domsta [Do], brought the earliest result on the boundedness of P_S on non-uniform meshes, namely on dyadic meshes.

Mityagin [Mi] developed rather general approach oriented on the case of $m = k/2-1$, but could prove only the cases of strictly geometric meshes for $k = 4, 6$ (covered later in [J1]).

1.2. Mesh-independent bounds. All the interest, and all the difficulties are concerned, of course, with mesh-independent estimates. At present there is no visible general approach which could give some notable advances. However, there exist three approaches which have brought some positive if even modest mesh-independent results.

1.2a. Total positivity of a relative Gramian. Proved the case of parabolic splines (already mentioned)

$$\sup_{\Delta} \|P_{S_3(\Delta)}\|_\infty < c, \qquad \text{de Boor [B1,B5]}; \tag{1.4}$$

and the case of C-splines ($m = 0$) of any order k:

$$\sup_{\Delta} \|P_{S_{k,0}(\Delta)}\|_\infty < c_k, \qquad \text{de Boor [B3]}. \tag{1.5}$$

Also the case of cubic splines ($k = 4$) was announced to be proved [B5].

1.2b. Matrix analysis of a spline-interpolation problem. Also proved the case of C-splines:

$$\sup_{\Delta} \|P_{S_{k,0}(\Delta)}\|_\infty < c_k, \qquad \text{Zmatrakov, Subbotin [ZS]}. \tag{1.6}$$

1.2c. Exponential decay of null-splines. Proved the L_p-boundedness with p from a small neighbourhood of 2:

$$\sup_{\Delta} \|P_S\|_p < c(p,k), \quad p \in (2-\epsilon'_k, 2+\epsilon''_k), \qquad \text{author [Sh]}. \tag{1.7}$$

Theoretically, all these methods are applicable to the general case. But all of them are based on exact analysis of some exact arithmetic expressions, and complexity of this arithmetics beyond the above cases makes the general use of these methods rather problematic.

However, the list of particular cases is so short, that any new one seems not to be superfluous.

In [Sh, §7] we pointed out that, theoretically, some constants related to the value of $\|P_S\|_\infty$ could be computed as eigenvalues of some matrices of the relatively small order $(2m+2)\times(2m+2)$. Our short experiments at that time had shown that arithmetics of the entries gives no chances for such a computing by hands.

The goal of our present work was to find whether the MAPLE tools could change something. And they did.

Our main result is

Theorem 1. *For $m = 1, k \geq 5$ and $m = 2, k \geq 10$ the L_∞-norm of the L_2-projector onto C^m-splines is bounded independently of Δ:*

$$\sup_\Delta \|P_{S_{k,m}(\Delta)}\|_\infty < c_k, \qquad m = 1,\ k \geq 5; \quad m = 2,\ k \geq 10. \tag{1.8}$$

Since the case $m = 1$, $k = 3$ was proved by de Boor [B1,B5], and the case $k = 4$ was announced to be proved [B5], the case $m = 1$ is completely positively solved (up to a certain uncertainty for $k = 4$).

There is a MAPLE evidence that the result holds for all (small) m, namely

$$\|P_{S_{k,m}(\Delta)}\|_\infty \leq c_k, \qquad k \geq (m+1)^2 + 1,$$

i.e., for the orthoprojectors onto splines with relatively low smoothness. We checked it for

$$m = 3,\ k = 17, 18; \quad m = 4,\ k = 26, 27.$$

In our considerations we restricted ourselves with the case

$$m < k/2 - 1,$$

but such a restriction is not principal, it only simplifies the pre- and forthcoming analysis. So, one may expect that for relatively small m the MAPLE tools could help in proving the case of C^m-splines completely, i.e. for all $k \geq m + 2$.

Remark. There are neither computational, nor theoretical arguments for that the conjecture will not fail for some k, m. So far, it is verified to be true only for particular k, m with $m \leq 4$. In this respect, it is worthwhile to mention another de Boor's conjecture which was also concerned with splines of arbitrary order k, and which was shown [J2] to be false for $k \geq 20$(!). But it is also worthwhile to mention that a modest award will be offered for a counterexample as well (for details see [B2]).

2. Organisation of the paper

1) In §3 we remind that

$$\|P_S\|_\infty \ \overset{k}{\sim}\ \sup_j \|N_j^*\|_1,$$

where N_j^* are the elements of the basis biorthogonal to the basis of the L_∞-normalized B-splines $\{N_j\}$.

2) In §4 we prove that for each $N_j^* \in S_k^l(\Delta)$ there exist two subsets $\Delta_\nu \subset \Delta$ and two null-splines

$$s_\nu \in \tilde{S}_{2k}^l(\Delta_\nu), \quad s_1^{(r)}(a) = 0, \quad s_2^{(r)}(b) = 0, \quad r = \overline{0, k-1}. \tag{2.1}$$

such that

$$s_\nu^{(k)}(x) = N_j^*(x), \quad x \in [a, b]. \tag{2.2}$$

3) In §§5-6 we prove that the $L_2^{(k)}$-norms of any null-spline with null boundary condition (2.1) at the left endpoint a satisfy the inequality

$$\|s^{(k)}\|^2_{L_2[t_0,t_{i-1}]} + \|s^{(k)}\|^2_{L_2[t_0,t_i]} \leq \beta^2\, \|s^{(k)}\|^2_{L_2[t_i,t_{i+1}]}, \quad \beta^2 < \gamma^2(\rho_i + 1/\rho_i)^{-1}. \tag{2.3}$$

4) In §7 we prove that (2.3) implies

$$\|s_1^{(k)}\|_{L_1[I_{i-d}]} \le c_k \gamma^d \|s_1^{(k)}\|_{L_1[I_i]} \tag{2.4}$$

whence, by symmetry,

$$\|s_2^{(k)}\|_{L_1[I_{i+d}]} \le c_k \gamma^d \|s_2^{(k)}\|_{L_1[I_i]}. \tag{2.5}$$

5) In §8 we show that (2.2) − (2.5) implies

$$\|N_j^*\|_1 \le c_k \sum_{i=0}^N \gamma^i,$$

so that

$$\|P_{S_{k,m}(\Delta)}\|_\infty \le c_k', \quad \text{if} \quad \gamma_{k,m} < 1.$$

6) In §§9-11 we show that the values β providing (2.3) could be found as the largest eigenvalues of some special matrices $W_{k,m}$ of the order $(2m+2) \times (2m+2)$.

7) In §12 formulas for matrix A relating to W are given.

8) In §13, to illustrate this method, we calculate these eigenvalues in the simplest case $m = 0$, and obtain

$$\gamma_{k,0} < \frac{1}{k^2 - 1} < 1,$$

i.e., one more proof of (1.5)-(1.6). In this case all computations are done by hands.

9) In §§14-17 we prove that for $m = 1, 2$ also

$$\gamma_{k,1} < 1, \quad k \ge 5; \qquad \gamma_{k,2} < 1, \quad k \ge 10; \tag{2.6}$$

what finishes the proof of Theorem 1. §14 contains the statements leading to (2.6) in the case $m = 1$, §§15-16 presents their MAPLE proofs. The case $m = 2$ is given briefly in §17.

10) §18 contains a general observation for applicability of this scheme for arbitrary m and $k \ge (m+1)^2 + 1$, based on the cases $m \le 4$. We finish the article with a short conclusion.

Remark. The key-point of our method is inequality (2.3) which provides a mesh-independent decay of the $L_p^{(k)}$-norms of certain null-splines. Having such a decay one can prove the L_∞-boundedness of P_S in several ways. In [Sh] we did it constructing an orthonormal basis of $S_k(\Delta)$ as the k-th derivatives of some null-splines. However, that paper was dealt with smooth splines for which the corresponding inequalities have more complicated and weaker form than (2.3), so that we found it not possible to refer the reader therein. Therefore, we give here an independent proof, choosing this time a pass to biorthonormal spline bases.

3. Reduction to biorthonormal bases

For a given mesh

$$\Delta = \{t_i\}_{i=0}^N = \{a = t_0 < t_1 < \ldots < t_N = b\},$$

and given k, l define the extended knot-sequence

$$\Delta = \{\tau_j\}_{j=0}^{N'}$$

as the sequence

$$\underbrace{t_0, \ldots, t_0}_{k}, \underbrace{t_1, \ldots, t_1}_{l}, \ldots \underbrace{t_{N-1}, \ldots, t_{N-1}}_{l}, \underbrace{t_N, \ldots, t_N}_{k}.$$

Let $\{N_j\}$ be the B-spline basis of $S_k^l(\Delta)$ forming a partition of unity,

$$\sum_j N_j \equiv 1, \quad N_j \le 0, \quad \mathrm{supp}\, N_j = (\tau_j, \tau_{j+k}),$$

and $\{N_j^*\}$ be the basis of S biorthogonal to $\{N_j\}$, i.e.,

$$(N_i, N_j^*) = \delta_{ij}.$$

The following lemma is known somehow.

Lemma 3.1.

$$\|P_S\|_\infty \overset{k}{\sim} \sup_j \|N_j^*\|_1.$$

Proof. We have

$$P_S(f) = \sum_j (N_j^*, f)\, N_j,$$

and due to the inequality [B4]

$$\|\sum a_j N_j\|_\infty \overset{k}{\sim} \|(a_j)\|_{l_\infty} = \sup_j |a_j|,$$

we obtain

$$\|P_S\|_\infty := \sup_{\|f\|_\infty = 1} \|P_S(f)\|_\infty \overset{k}{\sim} \sup_{\|f\|_\infty = 1} \sup_j |(N_j^*, f)| = \sup_j \|N_j^*\|_1.$$

4. Reduction to null-splines of the order $2k$

Notations. Let $\delta = \{\tau_i\}_{i=p'}^{q'}$ be a subset of extended Δ, such that

$$\delta = \underbrace{t_p, \ldots, t_p}_{l_p}, \underbrace{t_{p+1}, \ldots, t_{p+1}}_{l}, \ldots \underbrace{t_{q-1}, \ldots, t_{q-1}}_{l}, \underbrace{t_q, \ldots, t_q}_{l_q} \tag{4.1}$$

We denote by $l_i = l_i(\delta)$ the multiplicity of $t_i \in \delta$ in (4.1). Notice that $l_i = l$ for all t_i except the end-points of δ.

For two functions $f, g \in C^{k-1}[a,b]$, we write

$$f\big|_\delta = g\big|_\delta,$$

if

$$f^{(r)}\big|_{t_i} = g^{(r)}\big|_{t_i}, \quad r = \overline{0, l_i - 1}, \quad i = \overline{p, q}.$$

Now for any such δ we define the subspace of null-splines as

$$\tilde S_{2k}^l(\delta) = \{s : \ s \in S_{2k}^l(\Delta), \ s\big|_{\delta = 0}\}.$$

Lemma 4.1. *For $\Delta = \{\tau_i\}_{i=0}^{N'}$ and given j set*

$$\Delta_1 := \{\tau_i\}_{i=0}^{j+k-1}, \quad \Delta_2 := \{\tau_i\}_{i=j+1}^{N'}.$$

Then there exist two null-splines $s_\nu \in \tilde S_{2k}^l(\Delta_\nu)$, so that

$$s_1^{(r)}\big|_a = 0, \quad s_2^{(r)}\big|_b = 0, \quad r = \overline{0, k-1};$$

such that

$$s_\nu^{(k)}(x) = N_j^*(x), \quad x \in [a,b].$$

Proof. For a given j set

$$\psi(x) := \frac{1}{(k-1)!} \prod_{\mu=j+1}^{j+k-1} (x - \tau_\mu),$$

and define a function $\psi_1 \in C^{k-1}[a,b]$ by interpolating conditions

$$\psi_1\big|_{\Delta_1} = 0, \quad \psi_1\big|_{\Delta_2} = \psi\big|_{\Delta_2}.$$

By definition,

$$\psi_1\big|_{\Delta_1} = 0; \quad \psi_1^{(r)}\big|_a = 0, \quad r = \overline{0, k-1}. \tag{4.2}$$

C.de Boor [B4] proved that if a function g_j satisfies

$$g_j \in W_1^k[a,b], \quad g_j\big|_\Delta = \psi_1\big|_\Delta, \tag{4.3}$$

then

$$(N_i, g_j^{(k)}) = \delta_{ij}, \tag{4.4}$$

i.e., the family $\{g_j^{(k)}\}$ is biorthogonal to $\{N_j\}$.

Define $s_1 = s(\psi_1, \Delta)$ as the spline

$$s_1 \in S_k^l(\Delta), \quad s_1\big|_\Delta = \psi_1\big|_\Delta. \tag{4.5}$$

Such a spline s_1 exists and is unique.

Due to (4.3)-(4.5), $s_1^{(k)}$ is orthogonal to each B-spline but the j-th, and since it belongs to $S_k^l(\Delta)$,

$$s_1^{(k)}(x) = N_j^*(x), \quad x \in [a,b].$$

The null conditions for s_1 (in particular at a) follow from (4.2) and (4.5), because of $\Delta_1 \subset \Delta$.

To obtain s_2 we set

$$s_2(x) := s_1(x) - \psi(x), \quad x \in [a,b].$$

5. Representation of the $L_2^{(k)}$-norms of null-splines

For any spline s of degree $2k-1$ on the mesh Δ define

$$F(s,x) := \sum_{r=0}^{k-1} (-1)^r s^{(k-1-r)}(x) s^{(k+r)}(x). \tag{5.1}$$

This function is indefinite integral

$$F(s,x) = \int [s^{(k)}(t)]^2\, dt,$$

so that for $[c,d] \subset [t_i, t_{i+1}]$ we have

$$\int_c^d [s^{(k)}(x)]^2\, dt = F(s, d-0) - F(s, c+0). \tag{5.2}$$

If s has deficiency l, i.e. if $s \in C^{2k-1-l}$, then

$$s^{(k+r)}(t_i+0) \neq s^{(k+r)}(t_i-0), \quad r = \overline{k-1-l, k-1},$$

and the function $F(s,x)$ is not continiuos at t_i's. However, if s is a null-spline with multiplicity l, i.e., if

$$s^{(k-1-r)}(t_i) = 0, \quad r = \overline{k-1-l, k-1}, \quad i = \overline{0, N},$$

then this discontinuities disappears in the product (5.1), and $F(s,x)$ is continuous at t_i's. In this case (5.2) holds for any $[c,d] \subset [t_0, t_N]$.

Set

$$F(i) := F(s, t_i).$$

Then, if $s \in \tilde{S}^l_{2k}(\Delta)$, we have

$$F(i) := \sum_{r=0}^{k-1-l} (-1)^r s^{(k-1-r)}(t_i) s^{(k+r)}(t_i), \tag{5.3}$$

and

$$\|s^{(k)}\|^2_{L_2[t_i, t_{i+d}]} = F(i+d) - F(i).$$

6. The main inequality

Proposition 6.1. *Let $l > k/2$, and $s \in \tilde{S}^l_{2k}(\Delta_1)$, so that*

$$s^{(r)}\big|_{t_0} = 0, \quad r = \overline{0, k-1}. \tag{6.1}$$

Then there exists a constant $\gamma = \gamma_{k,m}$, such that for any triples

$$t_{i-1}, t_i, t_{i+1} \in \Delta_1, \quad l_{i-1} = l_i = l_{i+1} = l,$$

holds

$$\begin{gathered} \|s^{(k)}\|^2_{L_2[t_0, t_{i-1}]} + \|s^{(k)}\|^2_{L_2[t_0, t_i]} \le \beta^2 \, \|s^{(k)}\|^2_{L_2[t_i, t_{i+1}]}, \\ \beta^2 < \gamma^2 (\rho_i + 1/\rho_i)^{-1}, \quad \rho_i = h_i / h_{i-1}. \end{gathered} \tag{6.2}$$

Supplement 6.1. *If*

$$l_{i+1} < l = l_i = l_{i-1}, \quad \text{but} \quad l_i + l_{i+1} \ge k,$$

then (6.2) *also holds with some other $\gamma' = \gamma'_{k,m}$*

Proof. Ineq. (6.2) is equivalent to the following one

$$\|s^{(k)}\|^2_{L_2[t_0, t_i]} \le \frac{1}{2} \|s^{(k)}\|^2_{L_2[I_{i-1}]} + \frac{\beta^2}{2} \|s^{(k)}\|^2_{L_2[I_i]}. \tag{6.3}$$

We prove that a little bit stronger, than (6.3), inequality takes place, namely

$$\|s^{(k)}\|^2_{L_2[t_0, t_i]} \le \beta \|s^{(k)}\|_{L_2[I_{i-1}]} \|s^{(k)}\|_{L_2[I_i]} \tag{6.4}$$

Ineq.(6.4) implies (6.3), since $2\beta xy \le x^2 + \beta^2 y$.

Notice, that condition (6.1) implies

$$F(0) := F(s,x)\big|_{x=t_0} = 0,$$

so that for the left-hand side of (6.5) we have

$$\begin{aligned} \|s^{(k)}\|^2_{L_2[t_0, t_i]} &= F(i) - F(0) = F(i) \\ &= \textstyle\sum_{r=0}^{k-1-l} (-1)^r s^{(k-1-r)}(t_i) s^{(k+r)}(t_i), \end{aligned} \tag{6.5}$$

Since s is a null-spline with l_i-multiple zeros at t_i's, where $l_i = l \ge k/2$ by the condition, it has at least $l_i + l_{i+1} = 2l \ge k$ zeros on each interval $[t_i, t_{i+1}]$. For such a function, Rolle-type arguments provides

$$|s^{(k-1-r)}(t_i)| \le \begin{cases} \|s^{(k-1-r)}\|_{L_\infty[I_{i-1}]} & \le \; c_k h_{i-1}^{r+1/2} \|s^{(k)}\|_{L_2[I_{i-1}]}, \\ \|s^{(k-1-r)}\|_{L_\infty[I_i]} & \le \; c_k h_i^{r+1/2} \|s^{(k)}\|_{L_2[I_i]} \end{cases} \qquad r = \overline{0, k-1}. \tag{6.6}$$

Since, moreover, $s\big|_{[t_\nu, t_{\nu+1}]}$ is a polynomial, i.e., from a finite dimensional subspace, the norms equivalence gives

$$|s^{(k+r)}(t_i)| \le \begin{cases} \|s^{(k+r)}\|_{L_\infty[I_{i-1}]} & \le \; c_k h_{i-1}^{-r-1/2} \|s^{(k)}\|_{L_2[I_{i-1}]}, \\ \|s^{(k+r)}\|_{L_\infty[I_i]} & \le \; c_k h_i^{-r-1/2} \|s^{(k)}\|_{L_2[I_i]}, \end{cases} \qquad r = \overline{0, k-1}. \tag{6.7}$$

Setting

$$h = \min(h_{i-1}, h_i), \quad H = \max(h_{i-1}, h_i),$$

and choosing from (6.6),(6.7) the inequalities with h, H respectively, and substituting these estimates into (6.5) we obtain

$$\begin{aligned} \|s^{(k)}\|^2_{L_2[t_0,t_i]} \|s^{(k)}\|^{-1}_{L_2[I_{i-1}]} \|s^{(k)}\|^{-1}_{L_2[I_i]} &\le c_k \textstyle\sum_{r=0}^{k-1-l} (h/H)^{r+1/2} \\ &\le (m+1) c_k (h/H)^{1/2} \\ &\le 2^{1/2}(m+1) c_k (H/h + h/H)^{-1/2} \\ &= \gamma_{k,m} (h_{i-1}/h_i + h_i/h_{i-1})^{-1/2} \\ &= \gamma_{k,m} (\rho_i + 1/\rho_i)^{-1/2}, \end{aligned}$$

what proves (6.2) and the proposition.

The multiplicities of t_{i-1}, t_i, t_{i+1} as zeros of s were used only in (6.6) in the form

$$l_{i-1} + l_i \ge k, \quad l_i + l_{i+1} \ge k,$$

hence, the supplement.

Remark. The estimate (6.2), i.e., only with one interval involved in the right-hand side does not hold for multiplicities $l < k/2$. The necessary number ν of the intervals must satisfy the inequality $l(\nu+1) \ge k$.

7. $L_p^{(k)}$-estimates for null-splines

Corollary 7.1. *Let*

$$l > k/2, \quad s \in \tilde{S}^l_{2k}(\Delta_1).$$

Then with the same $\gamma = \gamma_{k,m}$ as in (6.2) for all $t_{i+1} \in \Delta_1$, such that $l_{i+1} + l_i \ge k$,

$$\|s^{(k)}\|_{L_p[I_{i-d}]} \le c_k \gamma^d \|s^{(k)}\|_{L_p[I_i]}, \quad 1 \le p \le \infty. \tag{7.1}$$

Proof. Evidently, (6.2) implies

$$\|s^{(k)}\|^2_{L_2[I_{i-1}]} \le \gamma^2 (\rho_i + 1/\rho_i)^{-1} \|s^{(k)}\|^2_{L_2[I_i]}. \tag{7.2}$$

Set

$$f_2(i) := \|s^{(k)}\|_{L_2[I_i]}, \quad f_p(i) := h_i^{1/p-1/2} f_2(i).$$

Since restriction of $s^{(k)}$ onto subintervals is a polynomial,

$$f_p(i) \;\; \overset{k}{\sim} \;\; \|s^{(k)}\|_{L_p[I_i]}. \tag{7.3}$$

In this notations (7.2) reads

$$f_2^2(i-1) < \gamma^2\,(\rho_i + 1/\rho_i)^{-1} f_2^2(i),$$

so that,

$$h_{i-1}^{2/p-1} f_2^2(i-1) < \gamma^2\,\frac{\rho_i^{2/p-1}}{\rho_i + 1/\rho_i} h_i^{2/p-1} f_2^2(i) = \gamma^2\,\frac{\rho_i^{2/p}}{\rho_i^2+1} h_i^{2/p-1} f_2^2(i).$$

For $p \in [1,\infty]$ and any ρ_i the factor with ρ_i is < 1, and we obtain

$$f_p(i-1) < \gamma f_p(i),$$

respectively,

$$f_p(i-d) < \gamma^d f_p(i),$$

and, due to (7.3), the estimate (7.1) is established.

8. L_1-norm estimates for N_j^*

Lemma 8.1. *For*

$$\Delta = \{t_i\} = \{\tau_{i'}\}, \quad \operatorname{supp} N_j = (\tau_j, \tau_{j+k}), \tag{8.1}$$

let j' be the index, such that

$$\tau_j = t_{j'}. \tag{8.2}$$

Then with the same γ as in (6.2),

$$\|N_j^*\|_{L_1[I_{j'-d}]} \le c_k \gamma^d \|N_j^*\|_{L_1[I_{j'}]} \le c_k \gamma^d \|N_j^*\|_{L_1[\tau_j,\tau_{j+k}]}. \tag{8.3}$$

Proof. Since the support of N_j is non-empty, it contains at least one interval I_i, so that by (8.1)$-$(8.2)

$$(t_{j'}, t_{j'+1}) \subset (\tau_j, \tau_{j+k}),$$

hence, the second inequality in (8.3).

To prove the first, set

$$\delta_1 = \{\tau_i\}_{i=j}^{j+k-1}, \quad \#\delta_1 = k.$$

Since $l \ge k/2$, this set contains at most 3 different t_i's, otherwise

$$\delta_1 = \{\underbrace{t_{j'},\ldots,t_{j'}}_{l_{j'}},\ \underbrace{t_{j'+1},\ldots,t_{j'+1}}_{l},\ \underbrace{t_{j'+2},\ldots,t_{j'+2}}_{l},\ t_{j'+3},\ldots\},$$

and

$$l_{j'} + 2l \ge 2l + 1 \ge k+1,$$

a contradiction.

Therefore, either

$$\delta_1 = \{\underbrace{t_{j'},\ldots,t_{j'}}_{l_{j'}(\delta_1)},\ \underbrace{t_{j'+1},\ldots,t_{j'+1}}_{l_{j'+1}(\delta_1)=l},\ \underbrace{t_{j'+2},\ldots,t_{j'+2}}_{l_{j'+2}(\delta_1)}\}, \tag{8.4}$$

or

$$\delta_1 = \{\underbrace{t_{j'},\ldots,t_{j'}}_{l_{j'}(\delta_1)},\ \underbrace{t_{j'+1},\ldots,t_{j'+1}}_{l_{j'+1}(\delta_1)}\} \tag{8.5}$$

with

$$l_{j'}(\delta_1) + l_{j'+1}(\delta_1) = k. \tag{8.6}$$

Consider

$$\Delta_1 = \{\tau_i\}_{i=0}^{j+k-1}.$$

Since $\delta_1 \subset \Delta_1$, in both cases

$$l_{j'}(\Delta_1) = l \geq l_{j'}(\delta_1), \quad l_{j'+1}(\Delta_1) = l_{j'+1}(\delta_1),$$

so that for the case (8.4) (since $2l \geq k$), as well as for the case (8.5) (due to (8.6)), we have

$$l_{j'}(\Delta_1) + l_{j'+1}(\Delta_1) \geq k, \tag{8.7}$$

By Corollary 7.1, for any null-spline $s \in \tilde{S}_{2k}^l(\Delta_1)$, condition (8.7) on j' gives

$$\|s^{(k)}\|_{L_p[I_{j'-d}]} \leq c_k \gamma^d \|s^{(k)}\|_{L_p[I_{j'}]}, \quad 1 \leq p \leq \infty.$$

By Lemma 4.1, there exists a null-spline $s_1 \in \tilde{S}_{2k}^l(\Delta_1)$, such that

$$N_j^*(x) = s_1^{(k)}(x),$$

and the lemma follows.

Lemma 8.2. *For any j, k,*

$$\|N_j^*\|_{L_1[\tau_j, \tau_{j+k}]} \leq c_k.$$

Proof. Set

$$\hat{N}_i = (k/\kappa_i)^{1/2} N_i, \quad \hat{N}_j^* = (\kappa_j/k)^{1/2} N_j^*,$$

that is $\hat{N}_j^*$ is biorthogonal to the basis of the L_2-normalized B-splines $\{\hat{N}_i\}$. C.de Boor [B3] proved that

$$\|\hat{N}_j^*\|_2 \leq c_k \|P_S\|_2 = c_k,$$

hence

$$\|N_j^*\|_{L_1[\tau_j, \tau_{j+k}]} = (k/\kappa_j)^{1/2} \|\hat{N}_j^*\|_{L_1[\tau_j, \tau_{j+k}]} \leq k^{1/2} \|\hat{N}_j^*\|_{L_2[\tau_j, \tau_{j+k}]} \leq c_k'.$$

Combining Lemmas 8.1-8.2 we obtain

$$\|N_j^*\|_{L_1[a,b]} \leq c_k \sum_{i=0}^N \gamma^i,$$

hence, the final

Lemma 8.3. *If for $l \geq k/2$,*

$$\gamma = \gamma_{k,m} < 1,$$

then

$$\|N_j^*\|_{L_1[a,b]} \leq c_k, \quad j = \overline{0, N'},$$

and, respectively,

$$\|P_{S_{k,m}(\Delta)}\|_\infty \leq c_k'.$$

9. Reduction to an eigenvalue problem

Condition (6.1) implies

$$F(0) := F(s,x)\big|_{x=t_0} = 0,$$

hence, if one rewrite (6.2) in terms of $F(i) = F(s,t_i)$, we should look for the best γ in the inequality

$$F(i-1) + F(i) \le \beta^2\,[F(i+1) - F(i)], \quad \beta^2 < \gamma^2(\rho_i + 1/\rho_i)^{-1}. \tag{9.1}$$

For a l-multiple null-spline s, define the vectors

$$\begin{aligned} x_i &= (x_i^{(l)}, \ldots, x_i^{(2k-1-l)}) \\ &= (x_i^{(k-1-m)}, \ldots, x_i^{(k+m)}) \in \mathbf{R}^{2m+2} \end{aligned}$$

where

$$x_i^{(r)} = s^{(r)}(t_i)/r!, \quad r = 0, \ldots, 2k-1. \tag{9.2}$$

Define then a square symmetric matrix T by the rule:

$$F(i) =: (Tx_i, x_i),$$

Then T is a mirror diagonal matrix with the elements

$$t_{j,2m+3-j} = \frac{1}{2}(k-m-2+j)!\,(k+m+1-j)!.$$

In order to express

$$F(i-1) = (Tx_{i-1}, x_{i-1}), \quad F(i+1) = (Tx_{i+1}, x_{i+1})$$

in terms of x_i too, introduce square matrices A_i, such that

$$x_{i+1} = A_i x_i, \quad x_{i-1} = A_{i-1}^{-1} x_i.$$

It is known that

$$A_i = A(h_i) = D(h_i) A D(1/h_i),$$

where $A = A(1)$ is a matrix with respect to the interval $[0,1]$, and

$$D(h) = \mathrm{diag}\left(1, h, h^2, \ldots, h^{2m+1}\right)$$

(or any other diagonal matrix with $d_{j+1,j+1}/d_{jj} = h$).

Also,

$$A_i^{-1} = A(-h_i) = D(-h_i) A D(-1/h_i),$$

in particular,

$$A^{-1} = C_0 A C_0, \quad C_0 = \mathrm{diag}\,(1, -1, 1, -1, \ldots).$$

Finally, set

$$B_i := A_i^{-1}.$$

In terms of quadratic forms Eq. (9.1) is written as

$$(TB_{i-1}x_i, B_{i-1}x_i) + (Tx_i, x_i) \le \beta^2\big((TA_i x_i, A_i x_i) - (Tx_i, x_i)\big),$$

or

$$\big((B_{i-1}^* T B_{i-1} + T)x_i, x_i\big) \le \beta^2\big((A_i^* T A_i - T)x_i, x_i\big).$$

For two symmetric square matrices U_0, V_0 with positive definite V_0, the exact constant β^2 in the inequality

$$(U_0x, x) \le \beta^2 (V_0 x, x)$$

is equal to the largest eigenvalue of the matrix $V_0^{-1}U_0$.

That is, for

$$U_0 = B_{i-1}^* T B_{i-1} + T, \quad V_0 = A_i^* T A_i - T, \tag{9.3}$$

we should find, whether the largest root $\lambda_{\max} =: \beta^2$ of the polynomial

$$p(\lambda) = |V_0^{-1}U_0 - \lambda E| = |V_0^{-1}| \cdot |U_0 - \lambda V_0|, \tag{9.4}$$

satisfies

$$\lambda_{\max} < \gamma^2(\rho + 1/\rho)^{-1}, \quad \gamma < 1.$$

10. Some simplifications

Here we will simplify the expressions in (9.3) – (9.4), in order not to deal with T and with transposed matrices A^*, B^*. Such simplifications seems not to be much necessary for the future MAPLE use, but they may be useful for theoretical analysis (and at least simplify the MAPLE programs).

10.1. We have

$$T^{-1}A^*T = CAC, \quad C = C^* = \text{diag}\,(\underbrace{\ldots, -1, 1}_{m+1}, \underbrace{1, -1, \ldots}_{m+1}). \tag{10.1}$$

Proof. Set

$$C_1 = \text{diag}\,(\underbrace{1, \ldots, 1}_{m+1}, \underbrace{-1, \ldots, -1}_{m+1}).$$

Then for any two null-splines $s_1, s_2 \in \tilde{S}_{2k}^l(\Delta)$, the vectors $x, y \in \mathbf{R}^{2m+2}$ of their derivatives

$$x_i^{(r)} = s_1^{(r)}(t_i)/r!, \quad y_i^{(r)} = s_2^{(r)}(t_i)/r!, \quad r = l, \ldots, 2k-1-l,$$

satisfy

$$(C_1 T x_i, y_i) = \sum_{r=l}^{2k-1-l} (-1)^r s_1^{(r)}(t_i) s_2^{(2k-1-r)}(t_i). \tag{10.2}$$

For two polynomials $f, g \in P_{2k}$ (of degree $\le 2k-1$), consider the function

$$\Phi(f, g; x) = \sum_{r=0}^{2k-1} (-1)^r f^{(r)}(x) g^{(2k-1-r)}(x).$$

We have

$$\Phi'(f, g; x) = 0,$$

therefore

$$\Phi(f, g; x) = c(f, g). \tag{10.3}$$

If s_1, s_2 are two null-splines on Δ, then $\Phi(s_1, s_2; x)$ is continuous in t_i's, and according to (10.3)

$$\Phi(s_1, s_2; x) = c(s_1, s_2), \quad x \in [a, b].$$

In particular,

$$\Phi(s_1, s_2; t_i) = \Phi(s_1, s_2; t_{i+1}). \tag{10.4}$$

But for $x = t_i$

$$\Phi(s_1, s_2; x) = \sum_{r=l}^{2k-1-l} (-1)^r s_1^{(r)}(x) s_2^{(2k-1-r)}(x),$$

and from (10.2),(10.4) we conclude that

$$(C_1 T x_i, y_i) = (C_1 T x_{i+1}, y_{i+1}) = (C_1 T A_i x_i, A_i y_i), \quad \forall x_i, y_i \in \mathbf{R}^{2m+2}.$$

This implies

$$C_1 T = A_i^* C_1 T A_i.$$

Since $C = C^*$, $T = T^*$, the transposition will give

$$T C_1 = A_i^* T C_1 A_i,$$

Hence

$$T^{-1} A_i^* T = C_1 A_i^{-1} C_1 = C_1 C_0 A_i C_0 C_1 = C A_i C,$$

with

$$C = \pm C_1 C_0 = \pm \,\mathrm{diag}\,(\underbrace{1, \ldots, 1}_{m+1}, \underbrace{-1, \ldots, -1}_{m+1})\,\mathrm{diag}\,(1, -1, \ldots).$$

In Eq. (10.1) we take for C the sign $(-1)^{m+1}$.

10.2. From (10.3) we obtain

$$\begin{aligned} T^{-1} B_i^* T B_i &= T^{-1} (A_i^{-1})^* T A_i^{-1} \\ &= C A_i^{-1} C A_i^{-1} \\ &= C D(h_i) A^{-1} D(1/h_i) C D(h_i) A^{-1} D(1/h_i) \\ &= D(h_i) C A^{-1} C A^{-1} D(1/h_i) \end{aligned}$$

that is

$$B_i^* T B_i = T D(h_i) C A^{-1} C A^{-1} D(1/h_i). \tag{10.5}$$

Similarly,

$$A_i^* T A_i = T D(h_i) C A C A D(1/h_i). \tag{10.6}$$

10.3. Basing on (10.5) – (10.6), from (9.3)-(9.4) we obtain

$$\begin{aligned} p(\lambda) &:= \left| B_{i-1}^* T B_{i-1} + T - \lambda \big(A_i^* T A_i - T\big) \right| \\ &= |T| \, \big| D(h_{i-1}) C A^{-1} C A^{-1} D(1/h_{i-1}) + E \\ &\qquad - \lambda \left(D(h_i) C A C A D(1/h_i) - E \right) \big| \\ &= \big| T D(h_i) C \big| \, \big| D(\rho_i) A^{-1} C A^{-1} D(1/\rho_i) + C \\ &\qquad - \lambda \left(A C A - C \right) \big| \, \big| D(1/h_i) \big|. \end{aligned}$$

That is

$$p(\lambda) = c_k \det (W),$$

with

$$U := D(\rho) A^{-1} C A^{-1} D(1/\rho) + C = D(\rho) C_0 A C A C_0 D(1/\rho) + C,$$

$$V := A C A - C, \quad W := U - \lambda V.$$

If for k, m given, with $m \le k/2 - 1$, the largest $\lambda = \lambda_{\max}$ for which $p(\lambda) = 0$ satisfies

$$\lambda_{\max} =: \beta^2 < \gamma^2 (\rho + 1/\rho)^{-1}, \quad 0 < \gamma = \gamma_{k,m} < 1,$$

then de Boor's conjecture for the case k, m is true:

$$\sup_{\Delta} \|P_{S_{k,m}(\Delta)}\|_\infty < c_k.$$

11. The algorithm

Step 1. Form the matrix A.

Step 2. Form the matrices

$$\begin{aligned} D_1 &:= \operatorname{diag}(1, \rho, \dots, \rho^{2m+1}), \\ D_2 &:= \operatorname{diag}(1, 1/\rho, \dots, 1/\rho^{2m+1}), \\ C_0 &:= \operatorname{diag}(1, -1, \dots, 1, -1), \\ C &:= \operatorname{diag}(\underbrace{\dots, -1, 1}_{m+1}, \underbrace{1, -1, \dots}_{m+1}). \end{aligned} \tag{11.1}$$

Step 3. Form the matrices

$$U := D_1 C_0 A C A C_0 D_2 + C, \quad V := ACA - C, \quad W := U - \lambda W. \tag{11.2}$$

Step 4. For the largest root $\lambda_{\max}$ of the equation

$$p(\lambda) = \det(W) = 0,$$

find whether the estimate

$$\lambda_{\max} < \gamma_{k,m}^2(\rho + 1/\rho), \quad \gamma^2 < 1,$$

takes place.

12. Formula for the matrix A

Lemma 12.1. *Let the polynomials $f_j \in P_{2k}[0,1]$ be defined as*

$$\begin{aligned} f_j^{(r)}(1) = f_j^{(r)}(0) &= 0, & r = \overline{0, l-1}; \\ f_j^{(l-1+r)}(0) &= \delta_{l-1+j, l-1+r}, & r = \overline{1, 2m+2}, \end{aligned} \tag{12.1}$$

and let

$$f_{ij} := f_j^{(l-1+i)}(1), \quad i, j = \overline{1, 2m+2};$$

Then

$$A = \{a_{ij}\}, \quad a_{ij} = \tfrac{(l-1+j)!}{(l-1+i)!)} f_{ij}, \quad i, j = \overline{1, 2m+2} \tag{12.2}$$

Proof. For any l-null polynomial (spline) $s \in P_{2k}[0,1]$, i.e., such that

$$s^{(r)}(0) = s^{(r)}(1) = 0, \quad r = \overline{0, l-1},$$

by definition (12.1) holds

$$s^{(l-1+i)}(1) = \sum_{j=1}^{2m+2} f_{ij}\, s^{(l-1+j)}(0). \tag{12.3}$$

In vector form, with normalisation (9.2), i.e.

$$x_i^{(r)} = \frac{1}{r!} s^{(r)}(t_i),$$

and with $t_0 = 0$, $t_1 = 1$ Eq. (12.3) reads

$$x_1^{(l-1+i)} = \textstyle\sum_{j=1}^{2m+2} \frac{(l-1+j)!}{(l-1+i)!} f_{ij}\, x_0^{(l-1+j)}.$$

Hence, for the matrix $A = \{a_{ij}\}$, such that

$$x_1 = A x_0$$

formula (12.2) takes place.

Lemma 12.2. *For*

$$A := A_{k,m} := \{a_{ij}\}_{i,j=1}^{2m+2}, \quad M = \min\{i,\, 2m+3-j\},$$

holds

$$a_{ij} = (-1)^{k-m-1} \sum_{\nu=1}^{M} (-1)^{\nu+1} \binom{k+m+1-j}{2m+3-j-\nu} \binom{k+m+1-\nu}{i-\nu}. \tag{12.4}$$

For the simplest entries, up to the sign, we obtain

$$a_{1,j} = \binom{k+m+1-j}{2m+2-j}, \quad a_{i,2m+2} = \binom{k+m}{i-1}, \quad i,j = \overline{1, 2m+2}, \tag{12.5}$$

in particular, for any k, m

$$a_{1,2m+2} = 1. \tag{12.6}$$

Proof. Consider the polynomial

$$p(x) := c_1 x^q \int_x^1 c_2 (1-y)^s y^t\, dy, \quad c_1 = 1/q!, \quad c_2 = \frac{(s+t+1)!}{s!\,t!},$$

so that

$$p^{(r)}(1) = 0, \quad r = \overline{0, s}; \qquad p^{(r)}(0) = \delta_{qr}, \quad r = \overline{0, q+t+1}.$$

We have

$$\int_x^1 = \sum_{\nu=1}^{t+1} \binom{s+t+1}{s+\nu} (1-x)^{s+\nu} x^{t+1-\nu},$$

thus,

$$p(x) = c_1 \sum_{\nu=1}^{t+1} \binom{s+t+1}{s+\nu} (1-x)^{s+\nu} x^{q+t+1-\nu}.$$

Further, for $x = 1$ and $i \ge \nu$,

$$\begin{aligned} \left[(1-x)^{s+\nu} x^{q+t+1-\nu}\right]^{(s+i)} &= \binom{s+i}{s+\nu} \left[(1-x)^{s+\nu}\right]^{(s+\nu)} \left[x^{q+t+1-\nu}\right]^{(i-\nu)} \\ &= \binom{s+i}{s+\nu} (-1)^{s+\nu} (s+\nu)!\, \frac{(q+t+1-\nu)!}{q+t+1-i)!} \end{aligned}$$

hence with $M = \min\{i, t+1\}$

$$\begin{aligned} p^{(s+i)}(1) &= \tfrac{1}{q!}\sum_{\nu=1}^{M} \tbinom{s+t+1}{s+\nu} \tfrac{(s+i)!}{(s+\nu)!\,(i-\nu)!}(-1)^{s+\nu}(s+\nu)!\,\tfrac{(q+t+1-\nu)!}{q+t+1-i)!} \\ &= (-1)^{s+1}\tfrac{(s+i)!}{q!}\sum_{\nu=1}^{M}(-1)^{\nu}\tbinom{s+t+1}{t+1-\nu}\tfrac{(q+t+1-\nu)!}{(q+t+1-i)!\,(i-\nu)!} \\ &= (-1)^{s+1}\tfrac{(s+i)!}{q!}\sum_{\nu=1}^{M}(-1)^{\nu}\tbinom{s+t+1}{t+1-\nu}\tbinom{q+t+1-\nu}{i-\nu} \end{aligned} \tag{12.7}$$

In our case, with

$$\begin{aligned} q = l-1+j &= k-m-2+j, \\ s = l-1 &= k-m-2, \\ t &= 2m+2-j, \end{aligned} \tag{12.8}$$

we have

$$p = f_j \in P_{2k}, \quad p^{(s+i)}(1) = f_{ij},$$

and

$$\tfrac{q!}{(s+i)!}p^{(s+i)}(1) = \tfrac{(l-1+j)!}{(l-1+i)!}f_{ij} = a_{ij}.$$

Thus, from the last equality in (12.7) we have

$$a_{ij} = (-1)^{s+1}\sum_{\nu=1}^{M}(-1)^{\nu+1}\binom{s+t+1}{t+1-\nu}\binom{q+t+1-\nu}{i-\nu}.$$

From (12.8)

$$\begin{aligned} s+t+1 &= k+m+1-j, \\ t+1 &= 2m+3-j, \\ q+t+1 &= k+m+1, \end{aligned}$$

so that, in terms of k, m, with $M = \min\{i, 2m+3-j\}$

$$a_{ij} = (-1)^{k-m-1}\sum_{\nu=1}^{M}(-1)^{\nu+1}\binom{k+m+1-j}{2m+3-j-\nu}\binom{k+m+1-\nu}{i-\nu}.$$

The lemma is proved.

The following MAPLE program computes the entries a_{ij} of $A_{k,m}$ for various $m = 0, 1, \ldots$ (in the example for $m = 1$) which are polynomials with respect to k of degree

$$d_{ij} = (i-1) + (2m+2-j).$$

```
> #++++++++++++++++++  Computation of A  +++++++++++++

> with(linalg):
> alias(R=binomial);
> m:=1;
> a:=(i,j) -> expand( sum(
  '(-1)^(n+1)*R(k+m+1-n,i-n)*R(k+m+1-j,2*m+3-j-n)',
  'n'=1..min(i,2*m+3-j) ) );
> A:=matrix(2*m+2,2*m+2,a);

> #++++++++++++++++++++++++++++++++++++++++++++++++++++
```

13. The case of C-splines ($m = 0$)

Step 1. To compute A by hands (as it was promised), we need only to calculate a_{21}, because from (12.5) – (12.6) up to the sign we obtain

$$a_{11} = a_{22} = k, \quad a_{12} = 1.$$

From (12.3) we have

$$a_{21} = \binom{k}{1}\binom{k}{1} - \binom{k}{0}\binom{k-1}{0} = k^2 - 1,$$

i.e.,

$$A = (-1)^{k-1} \begin{bmatrix} k & 1 \\ k^2 - 1 & k \end{bmatrix}$$

Step 2.

$$D_1 = \mathrm{diag}\,(1, \rho), \quad D_2 = \mathrm{diag}\,(1, 1/\rho),$$
$$C_0 = \mathrm{diag}\,(1, -1), \quad C = \mathrm{diag}\,(1, 1) = E.$$

Step 3.

$$\begin{aligned} V = A^2 - E &= 2 \begin{bmatrix} k^2 - 1 & k \\ k(k^2-1) & k^2 - 1 \end{bmatrix}, \\ U = D_1 C_0 (A^2 + E) C_0 D_2 &= 2 \begin{bmatrix} k^2 & -k/\rho \\ -k(k^2-1)\rho & k^2 \end{bmatrix}, \\ W = U - \lambda V &= 2 \begin{bmatrix} k^2 - \lambda(k^2-1) & -k(1/\rho + \lambda) \\ -k(k^2-1)(\rho+\lambda) & k^2 - \lambda(k^2-1) \end{bmatrix}. \end{aligned}$$

Step 4.

$$p(\lambda) = 4 \sum_{i=0}^{2} b_i \lambda^i,$$

with

$$b_0 > 0, \quad b_1, b_2 < 0,$$

namely

$$b_0 = k^2, \quad b_1 = -k^2(k^2-1)(\rho + 2 + 1/\rho), \quad b_2 = -(k^2 - 1).$$

Equation $p(\lambda) = 0$ implies

$$b_0 + b_1\lambda = -b_2\lambda^2 > 0,$$

i.e.

$$\lambda < -b_0/b_1 = \frac{1}{k^2-1}(\rho + 2 + 1/\rho)^{-1}.$$

Finally

$$\beta^2 := \lambda_{\max} < \gamma^2 (\rho + 1/\rho)^{-1}, \quad \gamma^2 = \frac{1}{k^2-1}. \tag{13.1}$$

14. The case of C^1-splines ($m = 1$)

Step 1. The MAPLE program gives Taylor expansion of the entries $a_{ij}(k)$ (which are polynomials with respect to k). We modified these expression to make them more compact.

$$A = (-1)^k \begin{bmatrix} \frac{k(k^2-1)}{6} & \frac{k(k-1)}{2} & k-1 & 1 \\ \frac{k(k+1)(k^2-4)}{6} & \frac{k(k^2-3)}{2} & k^2-2 & k+1 \\ \frac{(k+1)(k^2-3)(k^2-4)}{12} & \frac{(k^2-1)(k^2-4)}{4} & \frac{k(k^2-3)}{2} & \frac{k(k+1)}{2} \\ \frac{(k^2-4)[(k^2-3)(k^2-4)+6]}{36} & \frac{(k-1)(k^2-3)(k^2-4)}{12} & \frac{k(k-1)(k^2-4)}{6} & \frac{k(k^2-1)}{6} \end{bmatrix} \tag{14.1}$$

Examples. Here are the matrices $A = A_k$ for $k = 4, 5$

$$A_4 = \begin{bmatrix} 10 & 6 & 3 & 1 \\ 40 & 26 & 14 & 5 \\ 65 & 45 & 26 & 10 \\ 54 & 39 & 24 & 10 \end{bmatrix}, \ A_5 = - \begin{bmatrix} 20 & 10 & 4 & 1 \\ 105 & 55 & 23 & 6 \\ 231 & 126 & 55 & 15 \\ 273 & 154 & 70 & 20 \end{bmatrix}.$$

Step 2.

$$D_1 = \operatorname{diag}(1, \rho, \rho^2, \rho^3), \quad D_2 = \operatorname{diag}(1, 1/\rho, 1/\rho^2, 1/\rho^3),$$
$$C_0 = \operatorname{diag}(1, -1, 1, -1), \quad C = \operatorname{diag}(-1, 1, 1, -1).$$

Lemma 14.1. *Let $U, V, W \in \mathbf{R}^{4\times 4}$ be the matrices $(11.1) - (11.2)$, and $p \in P_5$ be the polynomial*

$$p(\lambda; k, \rho) := \det W = \sum_{i=0}^{4} b_i(k, \rho)\lambda^i. \tag{14.2}$$

Then for $k \geq 4$

$$b_0, b_2, b_3, b_4 > 0, \quad b_1 < 0.$$

As a consequence we obtain that, if $p(\lambda) = 0$, and $\lambda_{\max} > 0$, then

$$b_1\lambda_{\max} + b_2\lambda_{\max}^2 < 0,$$

whence

$$\lambda_{\max} < -\frac{b_1(k, \rho)}{b_2(k, \rho)}.$$

Lemma 14.2. *For $k \geq 5$, $m = 1$,*

$$\lambda_{\max} < -\frac{b_1(k, \rho)}{b_2(k, \rho)} < \gamma_{k,1}^2(\rho + 1/\rho)^{-1},$$

where

$$\gamma_{5,1}^2 = 11/12, \quad \gamma_{k,1}^2 = \frac{16}{k(k-2)}, \quad k \geq 6.$$

15. The MAPLE proof of Lemma 14.1

Since the entries of the matrices U, V are of the form

$$u_{ij} = p_{ij}(k)\rho^{i-j}, \quad v_{ij} = q_{ij}(k), \quad p_{ij}, q_{ij} \in P_{2m+4},$$

the coefficients $b_i(k,\rho)$ in (14.2) has the form

$$b_i(k,\rho) = \sum_{j=-\beta_i}^{\alpha_i} b_{ij}(k)\rho^j, \quad b_{ij} \in P_{d_i}.$$

For the concrete values $k \geq 4$ the MAPLE program shows that for every $i = \overline{0,4}$ the values b_{ij} are of the same sign for all $j = \overline{-\beta_i, \alpha_i}$. However, the Taylor expansion of $b_{ij}(k)$ with respect to k has both positive and negative coefficients. This is not a surprise, since we have the same picture, e.g., for the entries of A, see (14.1). Short experiments had shown that with $k = n+5$ the Taylor expansion of b_{ij} with respect to n is already strictly positive.

The following MAPLE programm gives for

$$k = n+5, \quad n \geq 0;$$

the coefficients $b_i(n,\rho)$ of the polynomial

$$p(\lambda; n+5, \rho) := \det W = \sum_{i=0}^{4} b_i(n,\rho)\lambda^i. \tag{15.1}$$

For these coefficients holds

$$b_i(n,\rho) = \rho^{-\alpha_i} \sum_{j=0}^{2\alpha_i} b_{ij}(n)\rho^j, \quad b_{ij} \in P_{d_i}, \tag{15.2}$$

i.e. $b_{ij}(n)$ are polynomials in n of degree $d_i - 1$.

The values α_i, d_i are the following:

i	0	1	2	3	4
α_i	0	3	4	3	0
$d_i - 1$	8	14	16	14	8

For every $i = \overline{0,4}$ the coefficients of the polynomials $b_{ij}(n)$ are of the same sign for all $j = \overline{0, d_i - 1}$, namely

$$\text{sign coeff}\,[b_{ij}] > 0, \quad i = 0,2,3,4; \quad \text{sign coeff}\,[b_{ij}] < 0, \quad i = 1.$$

Hence, for $n \geq 0$,

$$b_0, b_2, b_3, b_4 > 0, \quad b_1 < 0,$$

and Lemma 14.1 is proved. (As we had mentioned, for $k = 4$, i.e. for $n = -1$, the signs of the coefficients b_i are the same).

Remark. One can verify such a sign structure, looking directly at the MAPLE formulas for $b_i(n,\rho)$ in (15.1), i.e., without extra expansion (15.2). However each of these formulas occupies ca. one page, so we found it more convenient and safely for sign verification to make the further splitting.

```
> #++++++++++++   Taylor expansion of b_i, b_{ij}   +++++++

> with(linalg):
> with(powseries):
> alias(si=simplify); alias(R=binomial);

> k:=n+5; m:=1;
```

```
> #+++++++  Matrix A

> a:=(i,j) -> expand( sum(
  '(-1)^(n+1)*R(k+m+1-n,i-n)*R(k+m+1-j,2*m+3-j-n)',
  'n'=1..min(i,2*m+3-j) ) );
> A:=matrix(2*m+2,2*m+2,a);

> #+++++++  Matrices C,C0,D1,D2,U,V,W

> C:=diag(-1,1,1,-1); C0:=diag(-1,1,-1,1);
> D1:=diag(1,1/rho,1/rho^2,1/rho^3); D2:=diag(1,rho,rho^2,rho^3);
> V:=A&*C&*A-C; U:=D1&*C0&*A&*C&*A&*C0&*D2+C; W:=U-lambda*V;

> #+++++++  Polynomial p(lambda) = sum_i p(i)lambda^i

> p:=powpoly(det(W),lambda):
> d:=degree(det(W),n)+1;
> alpha:=degree(det(W),rho); beta:=ldegree(det(W),rho);

> #+++++++  Polynomials b_i(n,rho) = sum_j b_{ij}(n) rho^j

> b0:=taylor(si(p(0)),n,d);

> alpha1:=degree(p(1),rho); beta1:=ldegree(p(1),rho);
> b1:=powpoly(si(rho^3*p(1)),rho):
> b10:=taylor(b1(0),n,d);
> ...
> b16:=taylor(b1(6),n,d);

> alpha2:=degree(p(2),rho); beta2:=ldegree(p(2),rho);
> b2:=powpoly(si(rho^4*p(2)),rho):
> b20:=taylor(b2(0),n,d);
> ...
> b28:=taylor(b2(8),n,d);

> alpha3:=degree(p(3),rho); beta1:=ldegree(p(3),rho);
> b3:=powpoly(si(rho^3*p(3)),rho):
> b30:=taylor(b3(0),n,d);
> ...
> b36:=taylor(b3(6),n,d);

> b4:=taylor(p(4),n,d);

> #++++++++++++++++++++++++++++++++++++++++++++++++
```

16. The MAPLE proof of Lemma 14.2

We will show that for

$$k = n + 5, \quad n \geq 0,$$

and coefficients

$$b_1 := b_1(n, \rho), \quad b_2 := b_2(n, \rho)$$

of the expansion (15.1), with

$$\gamma^2 := \gamma_{k,1}^2 = \frac{16}{k(k-2)} := \frac{16}{(n+5)(n+3)}, \quad \text{or} \quad \gamma^2 := 11/12, \tag{16.1}$$

holds

$$-\frac{b_1}{b_2} < \gamma^2(\rho + 1/\rho)^{-1}. \tag{16.2}$$

Inequality (16.2) is equivalent to the following one

$$0 < \gamma^2 b_2 + (\rho + 1/\rho)b_1 =: q_1(n, \rho)$$

The functions $b_2(n, \rho)$, $b_1(n, \rho)$ have the lowest degree with respect to ρ equal to -4 and -3, respectively. With respect to n they are polynomials.

It would have been sufficient to have $\gamma^2 = 11/12$ for all k, but basing on the dependence of $\gamma_{k,0}$ on k in the case $m = 0$ (see (13.1)), we found it useful to have a similar quadratic decay for $\gamma_{k,1}$.

It turned out that $b_2(n, \rho)$ is divided by $(n + 5)(n + 3)$, i.e., the product γb_2, hence q_1, are also polynomials in n. In order to have q_1 as a polynomial both in ρ and n, we multiply it by ρ^4, and check the inequality

$$0 < \rho^4[\gamma b_2 + (\rho + 1/\rho)b_1] =: q(n, \rho). \tag{16.3}$$

The following MAPLE programm gives with γ from (16.2) the coefficients $c_i(n)$ of the polynomial

$$q := q(n, \rho) = \sum_{i=0}^{8} c_i(n)\rho^i.$$

For these coefficients holds

$$c_i \in P_{d_i}; \qquad d_i = 14;\ i = 0, 8; \quad d_i = 15,\ i = \overline{1, 7};$$

i.e. $c_i(n)$ are polynomials in n of degree $d_i - 1$.

For each $i = \overline{0, 8}$ the coefficients of the polynomials $c_i(n)$ are positive, i.e. (16.3), hence, (16.2) are true.

```
> #+++++++++++++   Coefficients c_i   +++++++++++

> ggamma:=16/((n+5)*(n+3));
> # ggamma:=11/12;
> q:=powpoly(si(rho^4*( ggamma*p(2) + (rho+1/rho)*p(1) )),rho):

> c0:=taylor(si(q(0)),n,d);
> ...
> c8:=taylor(si(q(8)),n,d);

> #+++++++++++++++++++++++++++++++++++++++++++++
```

Remark. For $k = 4$ we obtained

$$1 < 3/2 < \gamma_{4,1} < 2,$$

therefore no proof of boundedness $\|P_{S_{4,1}(\Delta)}\|_\infty$. This is, of course, no disproof as well. Such an estimate means only that there is no fast decrease of the $L_2^{(k)}$-norms of null-splines for $k = 4, m = 1$.

For $k = 3$ the inequality (6.2) does not hold at all, the right-hand side of (6.2), hence the matrix V_0 are only non-negative definite, and no bounds for $\gamma_{3,1}$ exist.

To cover the cases $k = 3, 4$ one may try to use instead of (6.2) an inequality involving on the right-hand side the L_2-norm over two intervals, i.e., over $[t_i, t_{i+2}]$.

17. The case of C^2-splines ($m = 2$)

Lemma 17.1. *For $m = 2$ let $U, V, W \in \mathbf{R}^{6\times 6}$ be the matrices (11.1) – (11.2), and $p \in P_7$ be the polynomial*

$$p(\lambda; k, \rho) := -\det W = \sum_{i=0}^{6} b_i(k,\rho)\lambda^i.$$

Then for $k \geq 6$

$$b_1, b_3, b_4, b_5, b_6 > 0, \quad b_0, b_2 < 0.$$

As a consequence we obtain that for $k \geq 6$ the largest root $\lambda_{\max} > 0$ of the equation $p(\lambda) = 0$ must satisfy at least one of the following inequalities

$$b_0 + b_1\lambda_{\max} < 0, \quad b_2\lambda_{\max}^2 + b_3\lambda_{\max}^3 < 0,$$

whence at least one of the following estimates holds:

$$\lambda_{\max} < -\frac{b_0(k,\rho)}{b_1(k,\rho)}, \quad \lambda_{\max} < -\frac{b_2(k,\rho)}{b_3(k,\rho)}.$$

Lemma 17.2. *For $k \geq 10$, $m = 2$,*

$$\lambda_{\max} < \max\left\{-\frac{b_0(k,\rho)}{b_1(k,\rho)}, -\frac{b_2(k,\rho)}{b_3(k,\rho)}\right\} < \gamma_{k,2}^2(\rho + 1/\rho)^{-1}, \tag{17.1}$$

where

$$\gamma_{10,2}^2 = 65/66, \quad \gamma_{11,2}^2 = 4/5, \quad \gamma_{k,2}^2 = \frac{81}{(k-1)(k-3)}, \quad k \geq 12. \tag{17.2}$$

Proof of Lemma 17.1. The MAPLE programm which is analogous to that in §15 gives for

$$k = n + 10, \quad n \geq 0;$$

the coefficients $b_i(n,\rho)$ of the polynomial

$$p(\lambda; n+10, \rho) := -\det W = \sum_{i=0}^{6} b_i(n,\rho)\lambda^i.$$

For these coefficients holds

$$b_i(n,\rho) = \rho^{-\alpha_i}\sum_{j=0}^{2\alpha_i} b_{ij}(n)\rho^j, \quad b_{ij} \in P_{d_i}.$$

The values α_i, d_i are the following:

i	0	1	2	3	4	5	6
α_i	0	5	8	9	8	5	0
$d_i - 1$	18	28	34	36	34	28	18

For every $i = \overline{0,6}$ the coefficients of the polynomials $b_{ij}(n)$ are of the same sign for all $j = \overline{0, d_i - 1}$, namely

$$\text{sign coeff}\,[b_{ij}] > 0, \quad i = 1,3,4,5,6; \qquad \text{sign coeff}\,[b_{ij}] < 0, \quad i = 0,2.$$

Hence,

$$b_1, b_3, b_4, b_5, b_6 > 0, \quad b_0, b_2 < 0,$$

and Lemma 17.1 is proved. (For $6 \leq k \leq 9$, the signs of the coefficients b_i are the same).

Proof of Lemma 17.2. Ineq. (17.1) is equivalent to the following two:

$$\begin{aligned} 0 &< \rho^5[\gamma b_1 + (\rho + 1/\rho) b_0] =: q_1(n,\rho), \\ 0 &< \rho^9[\gamma b_3 + (\rho + 1/\rho) b_2] =: q_2(n,\rho). \end{aligned} \tag{17.3}$$

The MAPLE programm gives with γ from (17.2) the coefficients $c_i(n), c_i'(n)$ of the polynomials

$$\begin{aligned} q_1 := q_1(n,\rho) &= \textstyle\sum_{i=0}^{10} c_i(n)\rho^i, \\ q_2 := q_2(n,\rho) &= \textstyle\sum_{i=0}^{18} c_i'(n)\rho^i. \end{aligned}$$

These coefficients are in turn polynomials in n,

$$c_i \in P_{26},\ \forall i; \qquad c_i' \in P_{34},\ i = 0, 18; \qquad c_i' \in P_{35},\ 1 \le i \le 17.$$

For each $i = \overline{0,18}$ the coefficients of the polynomials c_i, c_i' are positive, i.e. (17.3), hence, (17.1) are true.

18. The cases of C^m-splines, $m > 2$

We made some further computations with the MAPLE to see how the algorithm works for the cases $m = 3, 4$. They lead us to the following

Observation. Let $p \in P_{2m+3}$ be the polynomial

$$p(\lambda; k, \rho) := \det W = \sum_{i=0}^{2m+2} b_i(k,\rho)\lambda^i.$$

Then for $k \ge 2m+2$

$$\begin{aligned} \operatorname{sign} b_i &= -\operatorname{sign} b_{i+1}, & i &= \overline{0,m}, \\ \operatorname{sign} b_{m+1+i} &= \operatorname{sign} b_{m+1}, & i &= \overline{1,m+1}, \end{aligned}$$

so that the largest root of the equation $p(\lambda) = 0$ satisfy the inequality

$$\lambda_{\max} < \max_{0 \le i \le [m/2]} \left\{ -\frac{b_{m-2i}(k,\rho)}{b_{m+1-2i}(k,\rho)} \right\} < \gamma_{k,m}^2 (\rho + 1/\rho)^{-1}.$$

The maximum is attained for the ratio with $i = 0$, and

$$\gamma_{k,m} < 1 \qquad \Longleftrightarrow \qquad k \ge (m+1)^2 + 1.$$

This observation is confirmed by the cases

m	0	1	2	3	4
k	≥ 2	≥ 5	≥ 10	$17, 18$	$26, 27$

for which the above scheme has been realized, and for which de Boor's conjecture is proved thereby to be true.

19. Conclusion

It is clear that the above MAPLE algorithm may help to verify de Boor's conjecture only for particular values m. However, the results of our experiments give a hope that the general case of splines with high knot multiplicity could be proved by purely theoretical means.

This hope is based on the observation that the basic constant $\gamma_{k,m}$ from inequality (6.2) decays relatively fast with k growing, namely,

$$\gamma_{k,m} < c_m k^{-1}, \quad m = 0, 1, 2.$$

It looks quite probable that some general estimates like those used in the proof of (6.2) will bring the inequality

$$\gamma_{k,m} < 1 - \epsilon_m, \quad k > k_0(m).$$

This method will hardly give something for smooth C^m-splines of order k, if $k = O(m)$, so that original de Boor's conjecture still remains a mystery.

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