A NOTE ON THE LEAST CONSTANT IN LANDAU INEQUALITY ON A FINITE INTERVAL

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Abstract. In the Landau inequality on the unit interval

$$\|f^{(k)}\|_q \le \alpha\,\|f\|_p + \beta\,\|f^{(n)}\|_r$$

with $\|\cdot\|_s := \|\cdot\|_{L_s[0,1]}$, $1 \le p,q,r \le \infty$, $0 \le k < n$, we find the least value A_0 of the first constant α.

1. We are concerned here with the problem of Burenkov on sharp constants in Landau-type inequality on the unit interval

$$\|f^{(k)}\|_q \le \alpha\,\|f\|_p + \beta\,\|f^{(n)}\|_r \tag{1.1}$$

with

$$\|\cdot\|_s := \|\cdot\|_{L_s[0,1]}, \quad 1 \le p,q,r \le \infty, \quad 0 \le k < n.$$

Denote by Γ the set of all pairs (α,β) for which (1.1) holds for any $f \in W_r^n[0,1]$.

The general problem is to find the complete collection $G = \{(A,B)\}$ of sharp constants in (1.1) which are defined as

$$A \ge A_0 := \inf_{(\alpha,\beta)\in\Gamma} \alpha, \quad B := B(A) := \inf_{(A,\beta)\in\Gamma} \beta. \tag{1.2}$$

Here we define the least value of the first constant

$$A_0 = A_0(n,k,p,q,r). \tag{1.3}$$

2. The Landau-type inequalities in the additive form (1.1) were firstly studied by H.Cartan and Gorny for $p = q = r = \infty$. For arbitrary $p,q,r \in [1,\infty]$ they were obtained by Gabushin [2].

Burenkov [1] was first who was looking for the sharp constant (1.3) and the corresponding constant $B_0 := B(A_0)$. He proved that

$$A_0 = M_0, \quad k = n-1, \quad 1 \le p,q,r \le \infty,$$

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where

$$M_0 := M_0(n-1,k,p,q) := \sup_{P\in\pi_{n-1}} \frac{\|P^{(k)}\|_q}{\|P\|_p}, \tag{1.4}$$

is the best constant in the Markov-type inequality of different metrics for algebraic polynomials.

In [3],[4] it was shown that

$$A_0 = M_0, \quad 0<k<n, \quad p=q=r=\infty. \tag{1.5}$$

Moreover, the exact value for B_0 was also found. (In fact, Eq.(1.5) was proved much earlier by H.Cartan, though with a poor second constant.)

Here we give an elementary proof of the following

Theorem 1. *For any n,k,p,q,r*

$$A_0 = M_0.$$

3. Notice, that for all n,k,p,q,r the value M_0 provides the lower bound for A_0, i.e.

$$A_0 \ge M_0. \tag{1.6}$$

To see that, one can substitute in (1.1) instead of f an algebraic polynomial P_* of degree $n-1$ extremal for the Markov inequality (1.4).

Thus, it is enough to prove that (1.1) holds with $\alpha = M_0$ and some $\beta < \infty$ (the smaller is the better). We do it by finding an appropriate approximation to $f\in W_r^n$. Such a method was used by H.Cartan and Gorny, and was given in the most general form by S.B.Stechkin [5].

Proof of Theorem 1. Let $f\in W_r^n$, and let $P: W_r^n \to \pi_{n-1}$ be any projector from W_r^n onto the space π_{n-1} of algebraic polynomials of degree $n-1$. Then

$$\begin{aligned}\|f^{(k)}\|_q &\le \|P^{(k)}(f)\|_q + \|f^{(k)} - P^{(k)}(f)\|_q \\ &\le M_0\,\|P(f)\|_p + \|f^{(k)} - P^{(k)}(f)\|_q \\ &\le M_0\,\|f\|_p + M_0\,\|f-P(f)\|_p + \|f^{(k)} - P^{(k)}(f)\|_q.\end{aligned}$$

Set

$$\begin{aligned} L_\nu(P)_s &= \sup_{\|f^{(n)}\|_r\le 1} \|f^{(\nu)} - P^{(\nu)}(f)\|_s, \\ N_0(P) &= M_0 L_0(P)_p + L_k(P)_q, \\ N_0 &= \inf_P N_0(P).\end{aligned}$$

It is easy to show that $N_0<\infty$. For example, one can take as a P the Lagrange interpolating polynomial.

Hence,

$$\|f^{(k)}\|_q \le M_0\,\|f\|_p + N_0\,\|f^{(n)}\|_r,$$

that is

$$A_0 \le M_0, \quad B_0 \le N_0.$$

With respect to (1.6) this means that

$$A_0 = M_0,$$

which completes the proof.

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