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New inequalities from classical Sturm theorems

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Abstract

Inequalities satisfied by the zeros of the solutions of second-order hypergeometric equations are derived through a systematic use of Liouville transformations together with the application of classical Sturm theorems. This systematic study allows us to improve previously known inequalities and to extend their range of validity as well as to discover inequalities which appear to be new. Among other properties obtained, Szegő's bounds on the zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}(\cos\theta)$ for $|\alpha| < \frac{1}{2}$, $|\beta| < \frac{1}{2}$ are completed with results for the rest of parameter values, Grosjean's inequality (J. Approx. Theory 50 (1987) 84) on the zeros of Legendre polynomials is shown to be valid for Jacobi polynomials with $|\beta| \leqslant 1$, bounds on ratios of consecutive zeros of Gauss and confluent hypergeometric functions are derived as well as an inequality involving the geometric mean of zeros of Bessel functions.

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1. Introduction

Sturm theorems for second-order ODEs, in their different formulations, are well-known results from which a large variety of properties have been obtained (see for instance

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[5,7,10,12]). As a particular case of special relevance, bounds on the distances between consecutive zeros and convexity properties of the zeros of hypergeometric functions can be derived.

These results are usually based on adequate changes of both the dependent and the independent variables, which lead to a transformed differential equation which is simple to analyze.

For example, given a Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, the function

$$u(\theta) = \left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2} P_n^{(\alpha,\beta)}(\cos\theta) \tag{1}$$

satisfies a differential equation in normal form [12, p. 67]

$$\begin{aligned} &d^2u/d\theta^2 + A(\theta)u(\theta) = 0, \\ &A(\theta) = \left(n + \frac{\alpha+\beta+1}{2}\right)^2 + \frac{1/4-\alpha^2}{4\sin^2\dfrac{\theta}{2}} + \frac{1/4-\beta^2}{4\cos^2\dfrac{\theta}{2}}. \end{aligned} \tag{2}$$

When $|\alpha| < \frac{1}{2}$ and $|\beta| < \frac{1}{2}$ the coefficient $A(\theta)$ satisfies

$$A(\theta) > \left(n + \frac{\alpha+\beta+1}{2}\right)^2 \equiv A_M \tag{3}$$

and Sturm's comparison theorem provides the following bound on the distance between two consecutive zeros of $u(\theta)$ [12, p. 125]:

$$\theta_{k+1} - \theta_k < \frac{\pi}{\sqrt{A_M}} = \frac{\pi}{n + (\alpha+\beta+1)/2} \text{ when } |\alpha| < \frac{1}{2},\ |\beta| < \frac{1}{2}. \tag{4}$$

A similar analysis can be carried out, for instance, in the case of Laguerre polynomials, considering the function $v(x) = \exp(-x^2)x^{\alpha+1/2}L_n^{(\alpha)}(x^2)$. This gives a lower bound on the differences of square roots of consecutive zeros of Laguerre polynomials and also a bound on distances between consecutive zeros of Hermite polynomials $H_n(x)$ [12, p. 131]. The latter result comes from the fact that $H_n(\sqrt{x})$, $x > 0$, satisfies the differential equation for Laguerre polynomials with $\alpha = -\frac{1}{2}$. Another example is provided by the functions $\sqrt{x}C_\nu(x)$, $C_\nu(x)$ being a cylinder function (Bessel function), which satisfy differential equations in normal form suitable for the application of Sturm comparison theorem [13].

A question remains regarding this type of analysis: why make these changes of the dependent and independent variables and not others? In other words: what changes are amenable to a simple application of the Sturm theorems? In this paper, we perform a systematic study of Liouville transformations of the hypergeometric equations (Gauss and confluent) which lead to a simple analysis, in a sense to be made explicit later, of the monotonicity properties of the coefficient of the resulting differential equation (in normal form). The above-mentioned results for Jacobi, Laguerre and Hermite polynomials and for Bessel functions will be particular cases of the more general results provided by this systematic study.

Our analysis will also reveal convexity properties of the zeros and of simple functions of the zeros. For instance, we will see how Grosjean's convexity property [5] (see also [7]), for the zeros of Legendre polynomials

$$(1-x_k)^2 < (1-x_{k-1})(1-x_{k+1}) \tag{5}$$

also holds for the zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with $|\beta| \leqslant 1$ (Legendre polynomials being the particular case $\alpha = \beta = 0$) and in general for the zeros of any other solution of the corresponding differential equation in the interval $(0, 1)$.

In addition to these generalizations of previous results, inequalities which appear to be new can be obtained, like for instance bounds on ratios of consecutive zeros.

Our results will be valid for any non-trivial solution of the corresponding differential equation. We will restrict ourselves to real intervals where the coefficients of the differential equation are analytic and to those cases where the solutions of the differential equation have at least two zeros in that interval. This corresponds to the oscillatory situations studied in [3].

2. Methodology

We will consider the Sturm comparison and convexity theorems in the following form.

Theorem 1 (*Sturm*). *Let $y'' + A(x)y = 0$ be a second-order differential equation written in normal form, with $A(x)$ continuous in (a, b). Let $y(x)$ be a non-trivial solution of the differential equation in (a, b). Let $x_k < x_{k+1} < \ldots$ denote consecutive zeros of $y(x)$ in (a, b) arranged in increasing order. Then*

(1) *If there exists $A_M > 0$ such that $A(x) < A_M$ in (a, b) then*

$$\Delta x_k \equiv x_{k+1} - x_k > \frac{\pi}{\sqrt{A_M}}.$$

(2) *If there exists $A_m > 0$ such that $A(x) > A_m$ in (a, b) then*

$$\Delta x_k \equiv x_{k+1} - x_k < \frac{\pi}{\sqrt{A_m}}.$$

(3) *If $A(x)$ is strictly increasing in (a, b) then $\Delta^2 x_k \equiv x_{k+2} - 2x_{k+1} + x_k < 0$.*

(4) *If $A(x)$ is strictly decreasing in (a, b) then $\Delta^2 x_k \equiv x_{k+2} - 2x_{k+1} + x_k > 0$.*

Remark 2. An examination of the proof (Appendix A) shows that the first result still holds if there is one point in (a, b) where $A(x) = A_M$ and $A(x) < A_M$ elsewhere. For instance, we will find this case when $A(x)$ reaches a relative maximum in (a, b) and it is an absolute maximum in (a, b). The second result of the theorem can be generalized in the same way.

The third and fourth results of Theorem 1 are usually known as convexity theorem [7], which admits the following formulation.

Theorem 3 (*Sturm convexity theorem*). *Let $y''+A(x)y=0$ with $A(x)$ continuous in (a,b) and such that it may change sign in (a,b) at one point ($x=c$) at most. Let $A(x)$ be positive in an interval $I\subseteq(a,b)$ and, if $A(x)$ changes sign, let $A(x)<0$ in the rest of the interval (except at $x=c$).*

(1) *If $A(x)$ is strictly increasing in I then $\Delta^2 x_k\equiv x_{k+2}-2x_{k+1}+x_k<0$.*
(2) *If $A(x)$ is strictly decreasing in I then $\Delta^2 x_k\equiv x_{k+2}-2x_{k+1}+x_k>0$.*

These are well-known results. We provide a brief sketch of the proofs in Appendix A.

We will apply these theorems to confluent and Gauss hypergeometric functions, which are solutions of differential equations

$$y''+B(x)y'+A(x)y=0 \tag{6}$$

with one (confluent functions at $x=0$) or two finite singular regular points (Gauss hypergeometric function at $x=0$ and 1).

Our goal will be to obtain bounds on distances and convexity properties, either of the zeros or of simple functions of these zeros, which remain valid for all the zeros inside a given maximal interval of continuity of $B(x)$ and $A(x)$. In particular, we will focus on the intervals $(0,+\infty)$ for confluent functions and $(0,1)$ for Gauss hypergeometric functions; as we later discuss, properties in the rest of the maximal intervals can be obtained using linear transformations (Eqs. (17) and (18)).

The differential equations satisfied by the hypergeometric functions are not in normal form, but they can be transformed using a change of function, a change of variables or both. Given a solution $y(x)$ of a differential equation in standard form (Eq. (6)), the function $\tilde{y}(x)$ defined as

$$\tilde{y}(x)=\exp\left(\frac{1}{2}\int^x B(x)\right)y(x) \tag{7}$$

satisfies the equation

$$\tilde{y}''+\tilde{A}(x)\tilde{y}=0 \text{ with } \tilde{A}(x)=A-B'/2-B^2/4, \tag{8}$$

which is in the form suitable for the application of Theorem 1. In addition to these changes of the dependent variable, we can also consider changes of the independent variable $z=z(x)$, followed by a transformation to normal form. It is straightforward to check that given a function $y(x)$ which is a solution of Eq. (6) then the function $Y(z)$, with $Y(z(x))$ given by

$$Y(z(x))=\sqrt{z'(x)}\exp\left(\frac{1}{2}\int^x B(x)\right)y(x), \tag{9}$$

satisfies the equation in normal form

$$\ddot{Y}(z)+\Omega(z)Y(z)=0. \tag{10}$$

Here the dots mean differentiation with respect to z and

$$\Omega(z)=\dot{x}^2\tilde{A}(x(z))+\tfrac{1}{2}\{x,z\}, \tag{11}$$

where $\{x, z\}$ is the Schwarzian derivative of $x(z)$ with respect to z [8, p. 191]

$$\{x, z\} = -2\dot{x}^{1/2} \frac{d^2}{dz^2} \dot{x}^{-1/2} \tag{12}$$

and $\tilde{A}(x)$ is given by Eq. (8). This transformation of the differential equation is called a Liouville transformation, of crucial importance in the asymptotic analysis of second-order ODEs [8]. We can also consider $\Omega(z)$ as a function of x, which leads to the following expression:

$$\begin{aligned} \Omega(x) \equiv \Omega(z(x)) &= \frac{1}{z'(x)^2}(\tilde{A}(x) - \frac{1}{2}\{z, x\}) \\ &= \frac{1}{d(x)^2}\left(A(x) - \frac{B'(x)}{2} - \frac{B(x)^2}{4} + \frac{3d'(x)^2}{4d(x)^2} - \frac{d''(x)}{2d(x)}\right), \end{aligned} \tag{13}$$

where $\{z, x\}$ is the Schwarzian derivative of $z(x)$ with respect to x and $d(x) = z'(x)$.

The transformed function $Y(x) \equiv Y(z(x))$, Eq. (9), has the same zeros as $y(x)$ in (a, b) provided that $B(x)$ is continuous in (a, b). Besides, the equation is in the form suitable for the application of Sturm theorems, because $Y(z)$ satisfies (10).

We will use the freedom to choose $d(x)$ conveniently so that the problem becomes tractable in the sense that the monotonicity properties of $\Omega(z)$ are easily obtained. For this purpose, it is preferable to study the monotonicity properties of $\Omega(x)$ rather than those of $\Omega(z)$. Let us notice that $\Omega(x)$ and $\Omega(z)$ have the same monotonicity properties provided we consider changes of variable such that $z'(x) > 0$ (because $\Omega'(x) = \dot{\Omega}(z)z'(x)$). In addition, we introduce a further simplification of the problem by restricting the analysis to those changes of variable for which solving the equation $\Omega'(x) = 0$ is equivalent to solving a quadratic equation. Within these restrictions, we will perform a detailed study of the monotonicity of $\Omega(x)$ for the available changes of variable.

We will now consider separately the case of the differential equations satisfied by the hypergeometric functions ${}_2\mathrm{F}_1$, ${}_1\mathrm{F}_1$ and ${}_0\mathrm{F}_1$, starting from ${}_p\mathrm{F}_1$ $p = 2$ and decreasing p. This study includes the whole family of hypergeometric functions that satisfy second-order ODEs for real parameters. The case of the differential equation satisfied by the ${}_2\mathrm{F}_0$

$$x^2 y'' + [-1 + x(a + b + 1)]y' + ab\, y = 0, \tag{14}$$

need not be considered separately, because if $y(\gamma, \lambda, x)$ is a set of solutions of the confluent hypergeometric equation (${}_0\mathrm{F}_1(\gamma; \lambda; x)$ being one of the solutions), then $w(x) = |x|^{-a} y(a, 1 + a - b, -1/x)$, for $x > 0$ or $x < 0$, are solutions of Eq. (14). In other words, the properties of the zeros of solutions of Eq. (14) can be related to the properties of the zeros of confluent hypergeometric functions.

3. Gauss hypergeometric equation

We consider the hypergeometric equation, satisfied by the Gauss hypergeometric functions ${}_2\mathrm{F}_1(a, b; c; x)$

$$x(1 - x)\, y'' + [c - (a + b + 1)x]\, y' - ab\, y = 0 \tag{15}$$

with the restrictions on the parameters that allow for oscillatory solutions in $(0, 1)$ (see [3]), namely

$$a < 0,\ b > 1,\ c - a > 1,\ c - b < 0 \tag{16}$$

or, by symmetry, the same relations interchanging a and b.

Properties of the zeros in the other two maximal intervals of continuity, $(-\infty, 0)$ and $(1, +\infty)$, again in the oscillatory case, can be derived from the properties of the zeros in $(0, 1)$ using linear transformations of the differential equations that map these other two intervals into $(0, 1)$ (see [2, vol. I, Chapter II]). Indeed, if we denote by $\psi(\gamma, \lambda; \mu, x)$ a set of solutions of the hypergeometric equation $x(1-x)y'' + (\mu - (\gamma + \lambda + 1)x)y' - \gamma\lambda y = 0$ in the interval $(0, 1)$, solutions in the other two intervals can be obtained by considering the fact that both

$$y(a, b; c; x) = (1 - x)^{-a}\psi(a, c - b; c; x/(x - 1)), \quad x < 0 \tag{17}$$

and

$$y(a, b; c; x) = x^{-a}\psi(a, a + 1 - c; a + b + 1 - c; 1 - 1/x), \quad x > 1 \tag{18}$$

are solutions of the hypergeometric differential equation $x(1-x)y'' + (c-(a+b+1)x)y' - aby = 0$.

Instead of the parameters a, b and c, we will normally use the real parameters

$$n = -a,\ \alpha = c - 1, \quad \beta = a + b - c, \tag{19}$$

which correspond to the standard notation for Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n}\, {}_2F_1(-n, n + \alpha + \beta + 1; \alpha + 1; (1 - x)/2). \tag{20}$$

The oscillatory conditions in the interval $(0, 1)$ (Eq. (16)) can be rephrased, in terms of the Jacobi parameters, as follows:

$$n > 0,\ n + \alpha + \beta > 0,\ n + \alpha > 0,\ n + \beta > 0. \tag{21}$$

Except in Theorem 11, in this section we always assume that n, α and β satisfy Eq. (21).

If we apply the transformations (7) and (8) to the hypergeometric differential equation (15) we arrive at an equation in normal form with

$$4\tilde{A}(x) = \frac{L^2 - \alpha^2 - \beta^2 + 1}{x(1 - x)} + \frac{1 - \alpha^2}{x^2} + \frac{1 - \beta^2}{(1 - x)^2}, \tag{22}$$

where

$$L = b - a = 2n + \alpha + \beta + 1. \tag{23}$$

The study of the monotonicity properties of $\tilde{A}(x)$ for all ranges of the parameters L, α and β, with the conditions (21) seems a difficult task, because it involves solving a cubic equation depending on three parameters in order to obtain the points were $\tilde{A}'(x) = 0$. We will consider the restriction before mentioned, that is, we will use changes of variable such that solving $\Omega'(x)=0$ is equivalent to solving a quadratic equation in the interval $(0,1)$ for any values of the parameters. This approach will allow us to obtain global inequalities which hold for all the zeros inside each interval of continuity of $\tilde{A}(x)$; classical inequalities [12], as well as new inequalities or generalizations of earlier inequalities [5], will be obtained in a systematic way.

For the Gauss hypergeometric equation there are several different types of changes of variables which provide such simple coefficients $\Omega(x)$. Looking at Eq. (13) it is easy to see that the term $\tilde{A}(x)/z'(x)^2$ will be simple for all parameters if the factor $1/z'(x)^2$ is proportional to certain powers of x and $1-x$, for instance

$$1/z'(x)^2 \propto x(1-x),\, x^2,\, (1-x)^2,\, x^2(1-x),\, x(1-x)^2,\, x^2(1-x)^2. \tag{24}$$

On the other hand, one can check that for these changes of variable the Schwarzian derivative term gives a contribution of the same type, and that the resulting $\Omega(x)$ is such that $\Omega'(x)=0$ is equivalent to a quadratic equation in $(0,1)$.

It is interesting to note that the changes of variable corresponding to Eq. (24) are those related to the different fixed point methods, stemming from first-order difference-differential equations (DDEs) available for the computation of the zeros of Gauss hypergeometric functions [3,4,9]. Interlacing properties between the zeros of contiguous hypergeometric functions are easily available from a simple analysis of these DDEs, as it was done in [11]. We will not explore here this type of properties.

The changes of variable described before (Eq. (24)) are not the only ones that lead to a simple $\Omega(x)$. In Appendix B we perform a more systematic analysis to prove that the changes of variable $z(x)$ such that

$$z'(x) \equiv d(x) = x^{p-1}(1-x)^{q-1},$$

where

$$p=0 \qquad \text{or} \qquad q=0 \qquad \text{or} \qquad p+q=1$$

are also valid. However, here we will only study in detail those changes of variable given by (24), which lead to inequalities in terms of elementary functions of the zeros.

In Appendix B we also show that interchanging the values of p and q is equivalent to interchanging α and β, and also x and $1-x$. Hence, it is enough to consider for instance $q \geqslant p$, and the analogous properties when $p \geqslant q$ follow immediately. Therefore, it is enough to take into account the cases $(p,q)=(\frac{1}{2},\frac{1}{2}),\,(0,1),\,(0,\frac{1}{2}),\,(0,0)$ in order to complete the analysis of the changes of variable given by Eq. (24).

3.1. The change $z(x)=\arccos(1-2x)$*: Szegő's bounds for Jacobi polynomials and related results*

For $p=q=\frac{1}{2}$, we can choose $z(x)=\arccos(1-2x)$, which maps the interval $(0,1)$ onto $(0,\pi)$. The new variable $z(x)$ is the angle θ in Eq. (1). We will use the notation $\theta(x)$ for the change of variables instead of $z(x)$. Applying the corresponding Liouville transformation we get

$$4\Omega(x) = L^2 - \frac{\alpha^2 - 1/4}{x} - \frac{\beta^2 - 1/4}{1-x}, \tag{25}$$

where

$$L = 2n + \alpha + \beta + 1. \tag{26}$$

The differential equation in normal form (Eq. (10)) corresponding to the function $\Omega(x(\theta))$ in Eq. (25), turns out to be the differential equation studied by Szegő [12] (Eq. (2)). Not surprisingly, the study of the monotonicity of $\Omega(x)$ leads to Szegő's bound when $|\alpha|, |\beta| \leqslant \frac{1}{2}$, in a slightly improved version (compare Eq. (4) with Theorem 4). It is straightforward to check that when the oscillatory conditions (Eq. (21)) are satisfied we have the following properties

(1) If $|\alpha| = |\beta| = \frac{1}{2}$, then $\Omega'(x) = 0$,
(2) otherwise:
  (a) If $|\alpha| \leqslant \frac{1}{2}$ and $|\beta| \leqslant \frac{1}{2}$, then $\Omega(x)$ has exactly one absolute extremum in $[0, 1]$ and it is a minimum.
  (b) If $|\alpha| \geqslant \frac{1}{2}$ and $|\beta| \geqslant \frac{1}{2}$, then $\Omega(x)$ has exactly one absolute extremum in $[0, 1]$ and it is a maximum.
  (c) If $|\alpha| \geqslant \frac{1}{2}$ and $|\beta| \leqslant \frac{1}{2}$, then $\Omega'(x) > 0$ in $(0, 1)$.
  (d) If $|\alpha| \leqslant \frac{1}{2}$ and $|\beta| \geqslant \frac{1}{2}$, then $\Omega'(x) < 0$ in $(0, 1)$.

In the cases where there is an extremum, it is reached at

$$x_e = \frac{\sqrt{|1/4 - \alpha^2|}}{\sqrt{|1/4 - \alpha^2|} + \sqrt{|1/4 - \beta^2|}} \tag{27}$$

and the value of $\Omega(x)$ at this point is

$$\Omega(x_e) = \frac{1}{4}\left[L^2 \pm \left(\sqrt{|1/4 - \alpha^2|} + \sqrt{|1/4 - \beta^2|}\right)^2\right] > 0, \tag{28}$$

where the $+$ sign applies when the extremum is a maximum and the $-$ sign when it is a minimum. Accordingly, the following relations are obtained in terms of $\theta(x)$.

Theorem 4. *Let n, α and β satisfy Eq.* (21). *Let x_k, $k = 1, \ldots, N$, $x_1 < x_2 < \cdots < x_N$, be the zeros of any solution of the hypergeometric equation in* $(0, 1)$ *and let* $\theta_k = \arccos(1 - 2x_k)$, $k = 1, \ldots, N$. *Then the following hold*:

(1) *If* $|\alpha| = |\beta| = \frac{1}{2}$, *then* $\Delta\theta_k = \frac{2\pi}{L}$,
(2) *otherwise*:
  (a) *If* $|\alpha| \leqslant \frac{1}{2}$ *and* $|\beta| \leqslant \frac{1}{2}$, *then* $\Delta\theta_k < \dfrac{2\pi}{\sqrt{L^2 + \left(\sqrt{1/4 - \alpha^2} + \sqrt{1/4 - \beta^2}\right)^2}}$
  (b) *If* $|\alpha| \geqslant \frac{1}{2}$ *and* $|\beta| \geqslant \frac{1}{2}$, *then* $\Delta\theta_k > \dfrac{2\pi}{\sqrt{L^2 - \left(\sqrt{\alpha^2 - 1/4} + \sqrt{\beta^2 - 1/4}\right)^2}}$.
  (c) *If* $|\alpha| \geqslant \frac{1}{2}$ *and* $|\beta| \leqslant \frac{1}{2}$, *then* $\Delta^2\theta_k < 0$.

(d) *If* $|\alpha| \leqslant \frac{1}{2}$ *and* $|\beta| \geqslant \frac{1}{2}$, *then* $\Delta^2\theta_k > 0$.

These results refine Szegő's bounds on distances between the θ-zeros of Jacobi polynomials, for $|\alpha| < \frac{1}{2}$ and $|\beta| < \frac{1}{2}$, and complete the range of possible parameters α and β.

We can obtain additional monotonicity results in the first two cases when we only consider zeros which lie on the same side with respect to the extremum x_{e} (either on the increasing or the decreasing side of $\Omega(x)$). Let us denote $\theta_{\mathrm{e}} = \arccos(1-2x_{\mathrm{e}})$ and $\mathrm{sign}(\theta-\theta_{\mathrm{e}}) = \mathrm{sign}(\theta_j - \theta_{\mathrm{e}})$ for $j = k, k+1, k+2$ (we assume that θ_j, $j = k, k+1, k+2$, lie on the same side with respect to θ_{e}). Then $\Delta^2\theta_k = \theta_{k+2} - 2\theta_{k+1} + \theta_k$ satisfies

$$\begin{array}{ll} (1)\ \text{If} & |\alpha| \leqslant \frac{1}{2} \text{ and } |\beta| \leqslant \frac{1}{2} \text{ (but not both equal to } \frac{1}{2}\text{) then} \\ & \mathrm{sign}(\theta-\theta_{\mathrm{e}})\Delta^2\theta_k < 0. \\ (2)\ \text{If} & |\alpha| \geqslant \frac{1}{2} \text{ and } |\beta| \geqslant \frac{1}{2} \text{ (but not both equal to } \frac{1}{2}\text{) then} \\ & \mathrm{sign}(\theta-\theta_{\mathrm{e}})\Delta^2\theta_k > 0. \end{array} \tag{29}$$

In the particular cases where $|\alpha| = |\beta|$, the possible extrema are reached at $x_{\mathrm{e}} = \frac{1}{2}$, that is $\theta_{\mathrm{e}} = \pi/2$, and Szegő's monotonicity results are obtained [12, p. 126, Theorem 6.3.3] as a particular case. In [1], a similar property, valid for $|\alpha| < \frac{1}{2}$ and $|\beta| \geqslant |\alpha|$, is proved; this is related to Case 4 in Theorem (4) and to Case 1 in Eq. (29). In the sequel, we will not insist on showing these partial monotonicity results and we will only consider bounds and inequalities corresponding to x-zeros (or simple functions of these zeros) which are satisfied in the whole interval $(0, 1)$.

3.2. *The change* $z(x) = \log(x)$*: generalization of Grosjean's inequality*

Taking $p = 0$, $q = 1$, we have the change $z(x) = \log(x)$. The corresponding $\Omega(x)$ function is

$$4\Omega(x) = -L^2 + \frac{L^2 - \alpha^2 + \beta^2 - 1}{1-x} + \frac{1-\beta^2}{(1-x)^2}, \tag{30}$$

where we see that the singularity at $x = 0$ has been absorbed by the new variable $z(x)$ and has disappeared from $\Omega(x)$.

Again, assuming that the oscillatory conditions (Eq. (21)) are fulfilled, we have the following monotonicity properties in $(0, 1)$:

(1) If $|\beta| \leqslant 1$, then $\Omega'(x) > 0$.

(2) If $|\beta| > 1$, then $\Omega(x)$ has only one absolute maximum, which is located at

$$0 < x_{\mathrm{e}} = \frac{L^2 - \alpha^2 - (\beta^2 - 1)}{L^2 - \alpha^2 + \beta^2 - 1} < 1, \tag{31}$$

where

$$\Omega(x_{\mathrm{e}}) = \frac{1}{16} \frac{[(L+\alpha)^2 - (\beta^2-1)][(L-\alpha)^2 - (\beta^2-1)]}{\beta^2 - 1} > 1. \tag{32}$$

Consequently, we have the following:

Theorem 5. *Let n, α and β satisfy Eq.* (21). *Let* $z(x) = \log(x)$. *Then the zeros of hypergeometric functions in* $(0, 1)$ *satisfy*

(1) *If* $|\beta| \leqslant 1$, *then* $\Delta^2 z_k < 0$. *Therefore* (*reversing the change of variable*) *the zeros of the hypergeometric function satisfy the inequality*

$$x_k^2 > x_{k-1}x_{k+1}. \tag{33}$$

(2) *If* $|\beta| > 1$, *then* $\Delta z_k > f(L, \alpha, \beta)$ *where*

$$f(L, \alpha, \beta) = 4\pi\sqrt{\frac{\beta^2 - 1}{[(L+\alpha)^2 - (\beta^2 - 1)][(L-\alpha)^2 - (\beta^2 - 1)]}} \tag{34}$$

or, in terms of the zeros of the hypergeometric function

$$\frac{x_{k+1}}{x_k} > \exp(f(L, \alpha, \beta)). \tag{35}$$

In terms of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, and denoting its zeros by $\tilde{x_k}$, we obtain:

Corollary 6. *Let n, α and β satisfy Eq.* (21). *Then the zeros of Jacobi polynomials satisfy*

$$(1)\ If\ |\beta| \leqslant 1\ then\ (1 - \tilde{x}_k)^2 > (1 - \tilde{x}_{k-1})(1 - \tilde{x}_{k+1}). \tag{36}$$

$$(2)\ If\ |\beta| > 1\ then\ \frac{1 - \tilde{x}_k}{1 - \tilde{x}_{k+1}} > \exp(f(L, \alpha, \beta)). \tag{37}$$

This result was proved by Grosjean [5] in the particular case of Legendre polynomials (see also [6]). Therefore, our result is a generalization of Grosjean's inequality to the case of Jacobi polynomials, and in fact to any solution of the corresponding differential equation.

Interchanging the values of p and q we have the change $z(x) = -\log(1 - x)$ and we get similar results, but with α and β interchanged, as well as x and $1 - x$, in Eqs. (33) and (35). In terms of the zeros of Jacobi polynomials, we get:

Corollary 7. *Let n, α and β satisfy Eq.* (21). *Then zeros of Jacobi polynomials satisfy*

$$(1)\ If\ |\alpha| \leqslant 1\ then\ (1 + \tilde{x}_k)^2 > (1 + \tilde{x}_{k-1})(1 + \tilde{x}_{k+1}). \tag{38}$$

$$(2)\ If\ |\alpha| > 1\ then\ \frac{1 + \tilde{x}_{k+1}}{1 + \tilde{x}_k} > \exp(f(L, \beta, \alpha)). \tag{39}$$

3.3. The change $z(x) = -\tanh^{-1}(\sqrt{1-x})$.

For $p = 0$ and $q = \frac{1}{2}$, we consider the following change of variables $z(x) = -\tanh^{-1}(\sqrt{1-x})$. After the corresponding Liouville transformation, the singularity at $x = 0$ disappears in $\Omega(x)$, namely

$$\Omega(x) = \beta^2 - \alpha^2 - \frac{1}{4} + \left(L^2 - 1/4\right)x - \frac{\beta^2 - 1/4}{1 - x}. \tag{40}$$

Again, always assuming that the oscillation conditions are fulfilled, it is easy to check the following monotonicity properties:

(1) If $|\beta| \leqslant \frac{1}{2}$ then $\Omega'(x) > 0$ in $(0, 1)$.

(2) If $|\beta| \geqslant \frac{1}{2}$ then $\Omega(x)$ has only one absolute maximum in $[0, 1]$, which is located at

$$0 < x_{\mathrm{e}} = 1 - \sqrt{\frac{\beta^2 - 1/4}{L^2 - 1/4}} \leqslant 1, \tag{41}$$

where

$$\Omega(x_{\mathrm{e}}) = \left(\sqrt{L^2 - 1/4} - \sqrt{\beta^2 - 1/4}\right)^2 - \alpha^2 > 0. \tag{42}$$

Consequently, we have that

Theorem 8. *Let n, α and β satisfy Eq.* (21) *and let $z(x) = -\tanh^{-1}(\sqrt{1-x})$. Then the zeros of hypergeometric functions in* $(0, 1)$ *satisfy the following inequalities*:

(1) *If $|\beta| \leqslant \frac{1}{2}$ then $\Delta^2 z_k < 0$, or, in terms of the zeros x_k of the hypergeometric function,*

$$\frac{x_{k+1}x_{k-1}}{x_k^2} < \frac{h(x_{k+1})h(x_{k-1})}{h(x_k)^2} \tag{43}$$

with

$$h(x) \equiv (1 + \sqrt{1-x})^2. \tag{44}$$

(2) *If $|\beta| \geqslant \frac{1}{2}$ then $\Delta z_k > p(L, \alpha, \beta)$, where*

$$p(L, \alpha, \beta) = \frac{\pi}{\sqrt{\left(\sqrt{L^2 - 1/4} - \sqrt{\beta^2 - 1/4}\right)^2 - \alpha^2}}. \tag{45}$$

This implies that

$$\frac{1 + \sqrt{1 - x_k}}{\sqrt{x_k}} \frac{\sqrt{x_{k+1}}}{1 + \sqrt{1 - x_{k+1}}} > \exp(p(L, \alpha, \beta)). \tag{46}$$

Similarly as before, if we consider the change of variables $z(x) = \tanh^{-1}(\sqrt{x})$, we have similar relations interchanging α and β, p and q, x and $1 - x$. Namely:

Corollary 9. *Let n, α and β satisfy Eq.* (21). *Then the zeros of hypergeometric functions in* $(0, 1)$ *satisfy*

(1) *If $|\alpha| \leqslant \frac{1}{2}$ then*

$$\frac{(1 - x_{k+1})(1 - x_{k-1})}{(1 - x_k)^2} < \frac{g(x_{k+1})g(x_{k-1})}{g(x_k)^2}, \tag{47}$$

where

$$g(x) \equiv (1 + \sqrt{x})^2. \tag{48}$$

(2) *If* $|\alpha|\geqslant\frac{1}{2}$ *then* $\Delta z_k > p(L,\beta,\alpha)$ *for* $z(x)=\tanh^{-1}(\sqrt{x})$, *this means that*

$$\frac{\sqrt{1-x_k}}{1+\sqrt{x_k}}\frac{1+\sqrt{x_{k+1}}}{\sqrt{1-x_{k+1}}} > \exp(p(L,\beta,\alpha)). \tag{49}$$

3.4. *The change* $z(x)=\log(x/(1-x))$

This change corresponds to the case $p=q=0$, and it treats the singularities at $x=0$ and 1 in the same way, as happened with the case $p=q=\frac{1}{2}$. This explains its invariance with respect to the replacement $x\leftrightarrow 1-x$. Both singularities are eliminated in $\Omega(x)$, which becomes

$$4\Omega(x)=-(L^2-1)x^2+(L^2+\alpha^2-\beta^2-1)x-\alpha^2. \tag{50}$$

This is a parabola with one absolute maximum at

$$0<x_{\mathrm{e}}<\frac{1}{2}\frac{L^2+\alpha^2-\beta^2-1}{L^2-1}<1, \tag{51}$$

where $\Omega(x)$ attains the value

$$\Omega(x_{\mathrm{e}})=\frac{1}{16}\frac{(L^2-1-(\alpha-\beta)^2)(L^2-1-(\alpha+\beta)^2)}{L^2-1}. \tag{52}$$

This result remains true for any set of values of the parameters consistent with oscillation. As a consequence of this we have

$$\Delta z_k > f(\beta,\alpha,L)=f(\alpha,\beta,L), \tag{53}$$

where f is defined in Eq. (34).

In terms of the zeros of the hypergeometric function, we have the following global bound.

Theorem 10. *The zeros of hypergeometric functions in* $(0,1)$ *satisfy*

$$\frac{1-x_k}{x_k}\frac{x_{k+1}}{1-x_{k+1}} > \exp(f(\alpha,\beta,L)) \tag{54}$$

for all values of the parameters consistent with oscillation (*Eq.* (21)).

In terms of the zeros of hypergeometric functions for $x<0$ this result can be expressed in an even simpler form. Indeed, using Eq. (17) it is straightforward to check the following:

Theorem 11. *Given a solution of the hypergeometric equation* (15) *which oscillates in* $(-\infty,0)$, *any two consecutive zeros in this interval satisfy*

$$\frac{x_{k+1}}{x_k} > \exp(f(c-1,a-b,c-b-a)) \tag{55}$$

for all the values of a, b and c consistent with oscillation in $(-\infty,0)$ (*Remark* 12).

For all the results in this section, except Theorem 11, we always consider that the parameters satisfy Eq. (21), which are the oscillatory conditions in $(0,1)$. For Theorem 11, the oscillatory conditions are given in the next remark.

Remark 12. For $x<0$ the oscillatory conditions are

$$\begin{aligned}&a<0,\ b<0,\ c-a>1,\ c-b>1 \text{ or}\\ &a>1,\ b>1,\ c-a<0,\ c-b<0.\end{aligned} \tag{56}$$

When these conditions are not satisfied, there are no solutions with two zeros in $(-\infty,0)$, see [3].

Going back to our original discussion in the interval $(0,1)$, we notice that Theorem 10 resembles a combination of the bound obtained in the case $p=0$ and $q=1$ (Eq. (35)) and the related bound for $p=1$ and $q=0$, which reads

$$\frac{1-x_k}{1-x_{k+1}} > \exp(f(L,\beta,\alpha)) \text{ for } |\alpha|>1. \tag{57}$$

Combining both we have, when $|\alpha|>1$ and $|\beta|>1$ simultaneously,

$$\frac{1-x_k}{x_k}\frac{x_{k+1}}{1-x_{k+1}} > \exp(f(L,\alpha,\beta)+f(L,\beta,\alpha)), \tag{58}$$

which is weaker than Eq. (54), because we impose no restriction on the parameters in Eq. (54) and also in an asymptotic sense, because $f(L,\alpha,\beta)/f(\alpha,\beta,L)\to 0$ as $L\to\infty$.

In Theorem 13, Eq. (54) is rephrased in terms of the zeros of Jacobi polynomials.

Theorem 13. *The zeros* (*in* $(0,1)$) *of Jacobi polynomials satisfy*

$$\frac{1-\tilde{x}_k}{1+\tilde{x}_k}\frac{1+\tilde{x}_{k+1}}{1-\tilde{x}_{k+1}} > \exp(f(\alpha,\beta,L)) \tag{59}$$

for all values of the parameters consistent with oscillation (*Eq.* (21)).

4. Kummer's confluent hypergeometric equation

The confluent hypergeometric equation

$$xy''+(c-x)y'-ay=0 \tag{60}$$

is satisfied by the confluent hypergeometric series ${}_1\mathrm{F}_1(a;c;x)$. We concentrate on the positive zeros of this or any other function which is a solution of Eq. (60). For the possible negative zeros of these functions the relations are similar because if $y_1(x)\equiv y(a;c;x)$ is a solution of Eq. (60) then $y_2(x)\equiv e^x y(c-a,c,-x)$ is a solution of the same equation too.

Instead of the parameters a and c, we will normally use

$$n=-a,\ \alpha=c-1. \tag{61}$$

This notation corresponds to the standard for Laguerre polynomials

$$L_n^{(\alpha)}(x) = \binom{n+\alpha}{\alpha} \, {}_1\mathrm{F}_1(-n; \alpha+1; x). \tag{62}$$

In terms of these parameters the oscillatory conditions [3] for the solutions of Eq. (60) in $(0, +\infty)$ are given by

$$n > 0, \; n + \alpha > 0. \tag{63}$$

Throughout this section, we assume that n and α satisfy Eq. (63).

Hermite polynomials are also related to the confluent hypergeometric equation because

$$H_n(x) = 2^n \mathrm{U}(-n/2; 1/2; x^2), \tag{64}$$

where $\mathrm{U}(a; c; x)$ is a solution of (60), namely the confluent hypergeometric function of the second kind.

Let us now study the differential equations in normal form after convenient changes of variable. As before, we write this transformed equation as

$$\ddot{Y}(z) + \Omega(z)Y(z) = 0 \tag{65}$$

and we study the monotonicity properties of $\Omega(x) \equiv \Omega(z(x))$.

If we transform the equation to normal form directly we obtain

$$4\Omega(x) = -1 + \frac{2L}{x} + \frac{1-\alpha^2}{x^2}, \tag{66}$$

where we now define

$$L = 2n + \alpha + 1. \tag{67}$$

This means that the trivial change $z(x) = x$ already provides information. Also, it is easy to see that other tractable changes of variable are $z(x) = \sqrt{x}$ and $z(x) = \log(x)$.

We can carry out a more general analysis of the admissible changes by considering those of the form $d(x) = z'(x) = x^{m-1}$ (and therefore $z(x) = x^m/m, m \neq 0$ and $z(x) = \log(x), m = 0$). For these changes we have

$$\Omega(x) = -\tfrac{1}{4}x^{-2m}(x^2 - 2Lx + \alpha^2 - m^2). \tag{68}$$

A careful analysis of this function for all values of the parameters reveals the following behaviour.

Lemma 14. *Let $\Omega(x)$ be given by Eq. (68) and suppose that the oscillatory conditions (Eq. (63)) are fulfilled. Let*

$$x_{\mathrm{e}} = \frac{m-1/2}{m-1}L - \frac{\sqrt{\Delta}}{m-1}, \tag{69}$$

where

$$\Delta = \left(m - \frac{1}{2}\right)^2 L^2 + m(1-m)(\alpha^2 - m^2). \tag{70}$$

Then, except for some cases when $|\alpha| < |m|$ *and* $m \in (0, \frac{1}{2})$ *simultaneously, one of the following situations takes place necessarily, regardless of the value of n*

(1) *Either* $\Omega(x)$ *has only one absolute extremum for* $x \geqslant 0$ *and it is a maximum, located at* x_e, *where* $\Omega(x_e) > 0$.

(2) *Or* $\Omega(x)$ *satisfies the conditions of Theorem 3 in* $(0, 1)$, $\Omega(x)$ *being strictly decreasing when it is positive.*

The situations (1) *and* (2) *take place for the following values*:

(I) *If* $|\alpha| > |m|$, *then the situation* (1) *takes place for all values of* $|\alpha|$.

(II) *If* $|\alpha| = |m|$ *then*:
 (a) *If* $m \leqslant \frac{1}{2}$, *then the situation* (1) *takes place.*
 (b) *If* $m \geqslant \frac{1}{2}$, *then the situation* (2) *takes place.*

(III) *If* $|\alpha| < |m|$ *then*
 (a) *If* $m < 0$, *then the situation* (1) *takes place.*
 (b) *If* $m > \frac{1}{2}$, *then the situation* (2) *takes place.*

In the previous lemma, it is understood that the corresponding limit should be taken when a given expression loses meaning. For instance, when $m = 1$ and $|\alpha| > |m|$ we understand that $x_e = \lim_{m \to 1} \dfrac{m - 1/2}{m-1} L - \dfrac{\sqrt{\Delta}}{m-1} = (\alpha^2 - 1)/L$. As a consequence of Lemma 14, Theorems 1 and 3 (see also Remark 2) we have

Theorem 15. *Let* $x_k, x_{k+1}, \ldots$, *with* $x_k < x_{k+1} < \cdots$, *be positive consecutive zeros of* $y(x)$, *which is a solution of the equation* $xy'' + (\alpha + 1 - x)y' + ny = 0$, *with* $n > 0$ *and* $n + \alpha > 0$. *Let*

$$\begin{aligned}
\delta_m x_k &= \Delta z(x_k) = z(x_{k+1}) - z(x_k) = \frac{x_{k+1}^m - x_k^m}{m}, \\
\delta_0 x_k &= \lim_{m \to 0} \frac{x_{k+1}^m - x_k^m}{m} = \log(x_{k+1}/x_k), \\
\delta_m^2 x_k &= \Delta^2 z(x_k) = (x_{k+2}^m - 2x_{k+1}^m + x_k^m)/m, \\
\delta_0^2 x_k &= \log(x_{k+2}) - 2\log(x_{k+1}) + \log(x_k).
\end{aligned} \tag{71}$$

Then:

(1) *If* $|\alpha| \leqslant |m|$ *and* $m \geqslant \frac{1}{2}$ (*simultaneously*) *then* $\delta_m^2 x_k > 0$.

(2) *If*:
 (a) $|\alpha| > |m|$ *or*
 (b) $|\alpha| = |m|$ *and* $m \leqslant \frac{1}{2}$ *or*
 (c) $|\alpha| < |m|$ *and* $m < 0$,
 then

$$\delta_m x_k > \frac{\pi}{\sqrt{\Omega(x_e)}} = 2\pi x_e^m \sqrt{\frac{1-m}{L x_e - \alpha^2 + m^2}}, \tag{72}$$

 where x_e *and* $\Omega(x_e)$ *are given by Eqs.* (69) *and* (68), *respectively.*

For $m=1$ the right-hand side of Eq. (72) should be understood as a limit

$$\delta_1 x_k > \lim_{m\to 1} \frac{\pi}{\sqrt{\Omega(x_e)}} = \pi\sqrt{\frac{\alpha^2-1}{L^2-(\alpha^2-1)}}. \tag{73}$$

We illustrate Theorem 15 with three simple examples, the cases $m=1, \frac{1}{2}$ and 0. The cases $m=\frac{1}{2}$ and 0 correspond to two linear difference-differential equations of first order satisfied by confluent hypergeometric functions. As commented in the case of Gauss hypergeometric functions, interlacing properties between the zeros of contiguous functions can be obtained by using Sturm methods as described in [11].

4.1. $m=1$

This corresponds to the trivial change of variable $z(x)=x$. In this case

$$4\Omega(x) = -1 + \frac{2L}{x} + \frac{1-\alpha^2}{x^2}, \tag{74}$$

which is strictly decreasing if $|\alpha|\leqslant 1$; the relative extremum for $|\alpha|>1$ in $(0,+\infty)$ is reached at $x_e = \dfrac{\alpha^2-1}{L}$ where $\Omega(x_e) = \dfrac{L^2-(\alpha^2-1)}{\alpha^2-1} > 0$.

Theorem 16. *The zeros of confluent hypergeometric functions in $(0,+\infty)$ and, in particular, the zeros of Laguerre polynomials $L_n^{(\alpha)}(x)$, satisfy the following properties under oscillatory conditions (Eq. (63))*

(1) *If $|\alpha|\leqslant 1$ then $\Delta^2 x_k > 0$, in other words*

$$x_k < (x_{k+1}+x_{k-1})/2. \tag{75}$$

(2) *If $|\alpha|>1$ then*

$$x_{k+1}-x_k > \pi\frac{\sqrt{\alpha^2-1}}{\sqrt{L^2-(\alpha^2-1)}}. \tag{76}$$

The zeros of Hermite polynomials $H_n(x)$ ($\alpha=-\frac{1}{2}$), $\tilde{x}_k$, satisfy

$$\tilde{x}_k^2 < (\tilde{x}_{k-1}^2+\tilde{x}_{k+1}^2)/2. \tag{77}$$

4.2. $m=\frac{1}{2}$

This corresponds to the change of variable $z(x)=2\sqrt{x}$. We have

$$\Omega(x) = -x + 2L - \frac{\alpha^2-1/4}{x}. \tag{78}$$

This function is monotonically decreasing for $|\alpha|\leqslant\frac{1}{2}$. For $|\alpha|>\frac{1}{2}$, it has only one local extremum for $x>0$, which is a maximum and it is reached at $x_e=\sqrt{\alpha^2-\frac{1}{4}}$, where

$\Omega(x_e) = 2(L - \sqrt{\alpha^2 - \frac{1}{4}})$. For $|\alpha| = \frac{1}{2}$ this value is also an upper bound for the function $\Omega(x)$, because its maximum value is reached at $x = 0$ in this case. Therefore the following holds.

Theorem 17. *The zeros of the confluent hypergeometric functions in* $(0, +\infty)$ *and, in particular, the zeros of Laguerre polynomials* $L_n^{(\alpha)}(x)$, *satisfy the following properties under oscillatory conditions* (*Eq.* (63))

(1) *If* $|\alpha| \leqslant \frac{1}{2}$ *then* $\Delta^2 \sqrt{x_k} > 0$, *that is*

$$\sqrt{x_k} < \frac{\sqrt{x_{k+1}} + \sqrt{x_{k-1}}}{2}. \tag{79}$$

(2) *If* $|\alpha| \geqslant \frac{1}{2}$ *then*

$$\Delta \sqrt{x_k} = \sqrt{x_{k+1}} - \sqrt{x_k} > \frac{\pi}{\sqrt{2(L - \sqrt{\alpha^2 - 1/4})}}. \tag{80}$$

The zeros of Hermite polynomials $\mathrm{H}_n(x)$ $(L = n + \frac{1}{2}$ *and* $\alpha = -\frac{1}{2})$ *satisfy the following two properties simultaneously*

$$\begin{aligned} x_k &< \frac{x_{k+1} + x_{k-1}}{2}, \\ x_{k+1} - x_k &> \frac{\pi}{\sqrt{2n+1}}. \end{aligned} \tag{81}$$

The bound for the distance between zeros of Hermite polynomials is given in [12, formula (6.31.21), p. 131].

4.3. $m = 0$

This corresponds to the change of variable $z(x) = \log(x)$. The singularities at $x = 0$ disappear from $\Omega(x)$, which becomes a parabola

$$4\Omega(x) = -x^2 + 2Lx - \alpha^2. \tag{82}$$

The maximum is reached at $x_e = L$, where $4\Omega(x_e) = L^2 - \alpha^2$. Therefore, the zeros of the confluent hypergeometric functions (like Laguerre polynomials) satisfy $\Delta \log(x) > \dfrac{2\pi}{\sqrt{L^2 - \alpha^2}}$.

Theorem 18. *The zeros of the confluent hypergeometric functions in* $(0, +\infty)$ *and, in particular, the zeros of Laguerre polynomials* $L_n^{(\alpha)}(x)$, *satisfy the following properties for any values of the parameters consistent with oscillation* (*Eq.* (63))

$$\frac{x_{k+1}}{x_k} > \exp\left(2\frac{\pi}{\sqrt{L^2 - \alpha^2}}\right). \tag{83}$$

The zeros of Hermite polynomials satisfy

$$\frac{\tilde{x}_{k+1}}{\tilde{x}_k} > \exp\left(\frac{\pi}{\sqrt{L^2-\alpha^2}}\right). \tag{84}$$

5. The confluent equation for the ${}_0F_1(;c;x)$ series: Bessel functions

The confluent hypergeometric equation

$$x^2y'' + (\nu+1)xy' + xy = 0 \tag{85}$$

has one solution that can be written as a hypergeometric series ${}_0F_1(;\nu+1;-x)$. The differential equation has oscillatory solutions only for $x > 0$ and the oscillatory solutions have an infinite number of zeros. We use $-x$ as argument and $c = \nu+1$ as parameter in the ${}_0F_1$ series because this notation provides a simple relation with Bessel functions: if $\phi(\nu, x)$ is a solution of (85), the function

$$y(x) = x^{\nu/2}\phi(;\nu;x^2/4) \tag{86}$$

is a solution of the Bessel equation

$$x^2y'' + xy' + (\nu^2 - x^2)y = 0 \tag{87}$$

for $x > 0$.

In particular, the regular Bessel function $J_\nu(x)$ is related to the ${}_0F_1(;\nu+1;-x)$ series.

Throughout this section we will express the results both in terms of the zeros of Bessel functions $c_{\nu,k}$ and the zeros of the solutions of (85).

With the changes of variable $z(x)$ such that $z'(x) = d(x) = x^{m-1}$ we obtain

$$\Omega(x) = \frac{4x+m^2-\nu^2}{4x^{2m}} \tag{88}$$

and, depending on the values of m and ν, all the cases described in Theorems 1 and 3 (or Remark 2) are possible. Namely the following holds:

Lemma 19. *Let $\Omega(x)$ given by Eq.* (88) *and let*

$$x_e = \frac{m(\nu^2-m^2)}{4(m-1/2)} \tag{89}$$

so that

$$\Omega(x_e) = \frac{1}{2mx_e^{2m-1}}. \tag{90}$$

Then the following hold:

(1) *If*

 (a) $|\nu| > |m|$ *and* $m \leqslant \frac{1}{2}$,

(b) *or* $|\nu| = |m| < \frac{1}{2}$,
(c) *or* $|\nu| < |m|$ *and* $m < 0$,
then the hypothesis of Theorem 3(1) *are satisfied*.

(2) *If*
(a) $|\nu| = |m| > \frac{1}{2}$,
(b) *or* $|\nu| < |m|$ *and* $m \geqslant \frac{1}{2}$,
then the hypothesis of Theorem 3(2) *are satisfied*

(3) *If* $|\nu| > |m|$ *and* $m > \frac{1}{2}$, *then* $\Omega(x)$ *reaches only one absolute extremum for* $x > 0$ *and it is a maximum located at* $x = x_{\mathrm{e}}$, *where* $\Omega(x_{\mathrm{e}}) > 0$. *Theorem* 1(1) (*with Remark* 2) *can be applied*.

(4) *If* $|\nu| < |m|$ *and* $m \in (0, \frac{1}{2})$, *then* $\Omega(x)$ *reaches only one absolute extremum for* $x > 0$ *and it is a minimum located at* $x = x_{\mathrm{e}}$, *where* $\Omega(x_{\mathrm{e}}) > 0$. *Theorem* 1(2) (*with Remark* 2) *can be applied*.

In addition, when $m = \frac{1}{2}$, *we have for* $x > 0$

(1) *If* $|\nu| > \frac{1}{2}$, *then* $\Omega'(x) > 0$ *and* $\Omega(x) < 1$.
(2) *If* $|\nu| = \frac{1}{2}$, *then* $\Omega(x) = 1$.
(3) *If* $|\nu| < \frac{1}{2}$, *then* $\Omega'(x) < 0$ *and* $\Omega(x) > 1$.

Then, using these results we have the following theorem.

Theorem 20. *Let* $x_k, x_{k+1}, \ldots$, *with* $x_k < x_{k+1} < \cdots$, *be positive consecutive zeros of solutions of* $x^2y'' + (\nu+1)y' + xy = 0$. *Let* $\delta_m x_k$ *and* $\delta_m^2 x_k$ *be as in Eq.* (71). *Then the following hold*:

(1) *If*
(a) $|\nu| > |m|$ *and* $m \leqslant \frac{1}{2}$,
(b) *or* $|\nu| = |m|$ *and* $m < \frac{1}{2}$,
(c) *or* $|\nu| < |m|$ *and* $m < 0$,
then $\delta_m^2 x_k < 0$.

(2) *If*
(a) $|\nu| = |m|$ *and* $m > \frac{1}{2}$,
(b) *or* $|\nu| < |m|$ *and* $m \geqslant \frac{1}{2}$,
then $\delta_m^2 x_k > 0$.

(3) *If* $|\nu| > |m|$ *and* $m \geqslant \frac{1}{2}$ *then* $\delta_m x_k > \pi/\sqrt{\Omega(x_{\mathrm{e}})}$.
(4) *If* $|\nu| = |m|$ *and* $m = \frac{1}{2}$ *then* $\delta_m x_k = \pi$.
(5) *If* $|\nu| < |m|$ *and* $m \in (0, \frac{1}{2}]$ *then* $\delta_m x_k < \pi/\sqrt{\Omega(x_{\mathrm{e}})}$.

where $x_{\mathrm{e}} = \dfrac{m}{4}\dfrac{\nu^2 - m^2}{\left(m - \dfrac{1}{2}\right)}$ *if* $m \neq \frac{1}{2}$ *and*

$$\Omega(x_{\mathrm{e}}) = \begin{cases} 1 & if\ m = \frac{1}{2}, \\ \dfrac{1}{2m x_{\mathrm{e}}^{2m-1}} & if\ m \neq \frac{1}{2}. \end{cases}$$

Relations between the zeros of Bessel functions can be obtained from Theorem 20 replacing x_k by $c_{\nu,k}^2/4$. When $m = \frac{1}{2}$ we obtain the following well-known result.

Theorem 21. *The zeros of Bessel functions $c_{\nu,k}$ satisfy*
(1) *If* $|\nu| > \frac{1}{2}$ *then* $c_{\nu,k+1} - c_{\nu,k} > \pi$.
(2) *If* $|\nu| = \frac{1}{2}$ *then* $c_{\nu,k+1} - c_{\nu,k} = \pi$.
(3) *If* $|\nu| < \frac{1}{2}$ *then* $c_{\nu,k+1} - c_{\nu,k} < \pi$.

When $m = 0$, $z(x) = \log(x)$ and $\delta_0^2 x_k = \log(x_{k+1}) - 2\log(x_k) + \log(x_{k-1}) < 0$ and then $x_k > \sqrt{x_{k-1}x_{k+1}}$. In terms of the zeros of Bessel functions this inequality can be written as follows:

Theorem 22. *Let $c_{\nu,k}$ be consecutive zeros of a Bessel function of order ν. Then*

$$c_{\nu,k} > \sqrt{c_{\nu,k-1}c_{\nu,k+1}}. \tag{91}$$

Using a variant of Sturm theorems, a related inequality was proved in [10], namely, that the extremum $c'_{\nu,k}$ between two consecutive zeros $c_{\nu,k}$ and $c_{\nu,k+1}$ satisfies $c'_{\nu,k} > \sqrt{c_{\nu,k}c_{\nu,k+1}}$.

6. Conclusions

We have developed a systematic study of transformations of second-order hypergeometric equations to normal form by means of Liouville transformations. We choose transformations such that the problem of computing the extrema or studying the monotonicity properties of the resulting coefficient reduces to solving a quadratic equation. Classical results on distances between zeros and convexity properties [12] are particular cases of the obtained properties. Other results, like the convexity property proved by Grosjean [5] for Legendre polynomials can be also obtained and generalized with our approach. In particular, Grosjean's inequality has been proved to be valid for Jacobi polynomials too. Other properties have also been derived, like bounds on ratios of consecutive zeros of Gauss and confluent hypergeometric functions and finally an inequality that involves the geometric mean of the zeros of Bessel functions.

Appendix A. Proof of Sturm theorems

The bounds on distances between consecutive zeros of Theorem 1 (and Remark 2) can be easily obtained using Sturm comparison theorem in the form given, for instance, in [13]. An even more direct proof can be found using the Ricatti equation associated to $y'' + A(x)y = 0$, similarly as was done in [10]. We prove the second result in Theorem 1 (also taking into account the comments in Remark 2) and the second result in Theorem 3 (which implies the fourth result in Theorem 1). The remaining results can be proved in an analogous way.

Let $x_k < x_{k+1}$ be consecutive zeros of $y(x)$, which is a non-trivial twice differentiable solution of $y'' + A(x)y = 0$ in (a, b), $A(x)$ being continuous in (a, b). Because $y(x)$ is non-trivial we have that necessarily $y'(x_k)y'(x_{k+1}) \neq 0$. Without loss of generality we can suppose that $y(x)$ is positive in (x_k, x_{k+1}). Then $y'(x_k) > 0$ and $y'(x_{k+1}) < 0$ and therefore the function

$$h(x) = -y'(x)/y(x) \tag{A.1}$$

satisfies $\lim_{x\to x_k^+} h(x) = -\infty$ and $\lim_{x\to x_{k+1}^-} h(x) = +\infty$. Furthermore $h(x)$ is differentiable in (x_k, x_{k+1}) and

$$h'(x) = A(x) + h(x)^2. \tag{A.2}$$

Assuming now that $A(x) > A_m > 0$ in (a, b) (with the exception of one point if Remark 2 is considered) it follows that $h' > A_m + h^2$ in (x_k, x_{k+1}) and then $g(x) \equiv h'(x)/(A_m + h(x)^2) - 1 > 0$. Therefore

$$\lim_{\varepsilon\to 0^+} \int_{x_k+\varepsilon}^{x_{k+1}-\varepsilon} g(x)\, dx > 0$$

so that

$$\frac{\pi}{\sqrt{A_m}} - (x_{k+1} - x_k) > 0.$$

This proves (2) of Theorem 1 (of course, this result remains valid in those situations described in Remark 2).

To prove the second result of Theorem 3 we consider the hypothesis of that theorem with $A'(x) < 0$ when $A(x) > 0$ in (a, b). With these hypothesis, it is obvious that if there exists $c \in (a, b)$ such that $A(x) < 0$ for every $x \in (c, b)$ then, for any non-trivial solution $y(x)$ in (a, b) there is at most one zero in $[c, b)$. This follows from the fact that $A(x) < 0$ in (c, b) and then $y(x)y''(x) > 0$ in (c, b). Let $x_k < x_{k+1} < x_{k+2}$ be consecutive zeros such that $A(x_k) > 0$ and $A(x_{k+1}) > 0$. Taking into account that $A(x) > A(x_{k+1})$ in (x_k, x_{k+1}), we have, similarly as before, that

$$\frac{\pi}{\sqrt{A(x_{k+1})}} > x_{k+1} - x_k \tag{A.3}$$

and, regardless of the sign of $A(x_{k+2})$, we have that $A(x) < A(x_{k+1})$ in (x_{k+1}, x_{k+2}) and therefore

$$\frac{\pi}{\sqrt{A(x_{k+1})}} < x_{k+2} - x_{k+1}. \tag{A.4}$$

Eqs. (A.3) and (A.4) imply that $\Delta^2 x_k = x_{k+2} - 2x_{k+1} + x_k > 0$, which proves the second result of Theorem 3.

Appendix B. General changes of variable for the Gauss hypergeometric equation

Starting from the Gauss hypergeometric equation (15) written in standard form (6), and considering a Liouville transformation with change of variable $z(x)$ such that $z'(x) =$

$x^{p-1}(1-x)^{q-1}$ we find (Eq. (13)) that

$$\Omega(x) = \frac{1}{4}x^{2(1-p)}(1-x)^{2(1-q)}\left(\frac{L^2-\alpha^2-\beta^2+1-2(p-1)(q-1)}{x(1-x)}\right.$$
$$\left. + \frac{p^2-\alpha^2}{x^2} + \frac{q^2-\beta^2}{(1-x)^2}\right). \tag{B.1}$$

Let us notice that interchanging the values of p and q is equivalent to interchanging α and β and x and $1-x$.

We want to obtain the values of p and q such that solving $P(x) = 0$ for $x \in (0, 1)$ is equivalent to solving a quadratic equation (or maybe a linear one), for any values of the parameters L, α and β. Taking the derivative, we find that it has the following structure

$$\Omega'(x) = x^{-2p-1}(1-x)^{-2q-1}P(x), \tag{B.2}$$

where $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ is a polynomial of degree 3 with coefficients depending on five parameters: L, α, β, p and q. Now, $\Omega'(x) = 0$ will be equivalent to a quadratic equation in $(0, 1)$ when $a_3 = 0$, when $P(0) = 0$ and then $P(x) = x(b_2x^2 + b_1x + b_0)$ or similarly when $P(1) = 0$. A lengthy but straightforward calculation gives

$$\begin{aligned} a_3 &= \frac{1}{2}(1-p-q)\left[L^2-(1-p-q)^2\right], \\ P(0) &= -\frac{1}{2}p(p^2-\alpha^2), \\ P(1) &= \frac{1}{2}q(q^2-\beta^2). \end{aligned} \tag{B.3}$$

Hence, the equivalence with a quadratic equation is true if and only if one of these conditions is satisfied:

$$\begin{aligned} &1.\ \ p+q=1, \\ &2.\ \ p=0, \\ &3.\ \ q=0, \end{aligned} \tag{B.4}$$

which confirms that the changes implied by Eq. (24) are indeed valid. The general changes induced by these conditions are themselves related to hypergeometric functions. Of course, given any valid change of variable, $z(x)$, $\tilde{z}(x) = K_1z(x) + K_2$, where K_1 and K_2 are constants is also valid and equivalent to $z(x)$ in the sense that they provide the same properties. As mentioned before, we always take $z(x)$ such that $z'(x) > 0$ for every x.

In the case $p > 0$ we can take as $z(x)$ the following incomplete beta function

$$\begin{aligned} z(x) &= \int_0^x t^{p-1}(1-t)^{q-1}\,dt = B_x(p,q) \\ &= \frac{x^p}{p}\,{}_2\mathrm{F}_1(1-q, p; p+1; x), \end{aligned} \tag{B.5}$$

and for $q > 0$ we may consider

$$z(x) = -B_{1-x}(q,p) = -\frac{(1-x)^q}{q}\,{}_2\mathrm{F}_1(1-p, q; q+1; 1-x). \tag{B.6}$$

These changes of variable do not make sense when $p = 0$ or $q = 0$, but the differences, $z(x_{k+1}) - z(x_k)$ do make sense in the limit $p \to 0$ (or $q \to 0$). Of course, these cases can be also considered separately.

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References

[1] S. Ahmed, A. Laforgia, M.E. Muldoon, On the spacing of the zeros of some classical orthogonal polynomials, J. London Math. Soc. 25 (2) (1982) 246–252.
[2] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Higher trascendental functions (Bateman Manuscript Project), McGraw-Hill, New York, 1953.
[3] A. Gil, W. Koepf, J. Segura, Numerical algorithms for the real zeros of hypergeometric functions, Numer. Algorithms 36 (2004) 113–134.
[4] A. Gil, J. Segura, Computing the zeros and turning points of solutions of second order homogeneous linear ODEs, SIAM J. Numer. Anal. 41 (3) (2003) 827–855.
[5] C.C. Grosjean, A property of the zeros of the Legendre polynomials, J. Approx. Theory 50 (1987) 84–88.
[6] A. Laforgia, Monotonicity Properties for the Zeros of Orthogonal Polynomials and Bessel Functions, Orthogonal Polynomials and Applications (Bar-le-Duc, 1984), Lecture Notes in Mathematics, vol. 1171, Springer, Berlin, 1985, pp. 267–277.
[7] A. Laforgia, M.E. Muldoon, Some consequences of the Sturm comparison theorem, Amer. Math. Monthly 93 (1986) 89–94.
[8] F.W.J. Olver, Asymptotics and Special Functions, AK Peters, Wellesley, 1997.
[9] J. Segura, The zeros of special functions from a fixed point method, SIAM J. Numer. Anal. 40 (2002) 114–133.
[10] J. Segura, On a conjecture regarding the extrema of Bessel functions and its generalization, J. Math. Anal. Appl. 280 (2003) 54–62.
[11] J. Segura, On the zeros and turning points of special functions, J. Comput. Appl. Math. 153 (2003) 433–440.
[12] G. Szegő, Orthogonal Polynomials, American Mathematical Society, Providence, Rhode Island, 1975.
[13] N.M. Temme, Special Functions, An Introduction to the Classical Functions of Mathematical Physics, Wiley, New York, 1996.