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Global Sturm inequalities for the real zeros of the solutions of the Gauss hypergeometric differential equation

Alfredo Deaño[a], Javier Segura[b,*]

[a] *Departamento de Matemáticas, U. Carlos III de Madrid, 28911-Leganés, Madrid, Spain*
[b] *Departamento de Matemáticas, Estadística y Computación. U. de Cantabria, 39005-Santander, Spain*

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Abstract

Liouville–Green transformations of the Gauss hypergeometric equation with changes of variable $z(x) = \int^x t^{p-1}(1-t)^{q-1}\,dt$ are considered. When $p+q=1$, $p=0$ or $q=0$ these transformations, together with the application of Sturm theorems, lead to properties satisfied by all the real zeros x_i of any of its solutions in the interval $(0,1)$. Global bounds on the differences $z(x_{k+1}) - z(x_k)$, $0 < x_k < x_{k+1} < 1$ being consecutive zeros, and monotonicity of these distances as a function of k can be obtained. We investigate the parameter ranges for which these two different Sturm-type properties are available. Classical results for Jacobi polynomials (Szegö's bounds, Grosjean's inequality) are particular cases of these more general properties. Similar properties are found for other values of p and q, particularly when $|p| = |\alpha|$ and $|q| = |\beta|$, α and β being the usual Jacobi parameters.

MSC: 33CXX; 34C10; 26D20

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1. Introduction

Different properties of the real zeros of the solutions of hypergeometric equations can be obtained by means of Liouville–Green (LG) transformations of the differential equation and the

* Corresponding author.
E-mail addresses: alfredo.deanho@math.uc3m.es (A. Deaño), javier.segura@unican.es (J. Segura).

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subsequent application of Sturm theorems. Among these, bounds on the distances between consecutive zeros and monotonicity properties of these distances can be obtained in the transformed variable $z(x)$.

In [1], Sturm properties of the solutions of second order hypergeometric equations were investigated by using sets of LG transformations that verify two basic requirements: first, the study of the transformed differential equation was analytically affordable; second, the results obtained from Sturm theorems provided global information on the zeros. The properties obtained were global in three ways: first, for fixed parameter values, they are valid for any solution of the differential equation; second, the Sturm properties apply to all real zeros in the given maximal interval of continuity of the coefficients of the ODE, which can be $(0, 1)$, $(-\infty, 0)$ or $(1, \infty)$ in the case of the Gauss hypergeometric equation; third, the properties hold independently on the parameter n (the degree for the polynomials cases), although the value of the bounds depends on n.

In the case of the Gauss hypergeometric equation, several particular transformations were considered, which led to a generalization of known results (Szegö's [5, pp. 124–126] and Grosjean's [3] inequalities), as well as to new inequalities satisfied by the real zeros of Gauss functions. As a particular case, this study provided global properties for all the real zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ in $(-1, 1)$.

In this paper, we investigate more general properties that have Szegö's [5, pp. 124–126] and Grosjean's [3] inequalities as particular cases and which are global in the three ways described above. In addition, the bounds will be optimal in a sense to be described later.

We consider the LG transformations of the hypergeometric equation with changes of variable

$$z(x) = \int^x t^{p-1}(1-t)^{q-1}\, dt \tag{1}$$

transforming the hypergeometric equation to an equation in normal form in the z variable:

$$\ddot{y}(z) + \Omega(z)y(z) = 0. \tag{2}$$

To this transformed equation, we apply Sturm's theorem.

The changes of variable (1), having derivative $z'(x) = x^{p-1}(1-x)^{q-1}$, are the natural generalization of the changes of variable considered in [1]. These changes do not introduce additional singularities; the only singularities in $\Omega(z(x))$ or $z(x)$ are at $x = 0, 1, +\infty$, which are the three singularities of the hypergeometric differential equation.

In [1] the particular cases $p, q = 0, \frac{1}{2}, 1$ (but not both equal to 1) were considered; $(p, q) = (\frac{1}{2}, \frac{1}{2})$ corresponds to Szegö's case and $(0, 1)$ or $(1, 0)$ to the (generalized) Grosjean's inequality. Here we investigate more general global inequalities which have these properties as particular cases.

Our analysis completes the picture of global properties obtainable from the Sturm comparison theorem that are valid for all the real zeros of any solution of the Gauss hypergeometric equation. We will concentrate on the interval $(0, 1)$; as discussed in [1], the corresponding Sturm properties for the other two maximal intervals of continuity follow from the properties in $(0, 1)$, taking into account linear transformations of the hypergeometric equation.

2. Methodology

The main result that we will consider is the Sturm comparison and convexity theorem. Denoting first and second order differences by

$$\begin{aligned}\Delta z_k &= z_{k+1} - z_k,\\ \Delta^2 z_k &= \Delta z_{k+1} - \Delta z_k = z_{k+2} - 2z_{k+1} + z_k,\end{aligned} \tag{3}$$

$z_k < z_{k+1} < z_{k+2}$, we can enunciate the Sturm theorem in the following form [1].

Theorem 1 (*Sturm*). *Let us consider a second order ordinary differential equation in normal form*:

$$\ddot{y}(z) + \Omega(z) y(z) = 0, \tag{4}$$

where $\Omega(z)$ is continuous in a given interval I. Let $y(z)$ be a nontrivial solution of this equation in I, and $z_k < z_{k+1} < z_{k+2}$ consecutive zeros of $y(z)$ in this interval.

- *If there exists $z_M \in I$ such that $\Omega(z) < \Omega(z_M)$ for all z in I different from z_M, then*: $\Delta z_k > \frac{\pi}{\sqrt{\Omega(z_M)}}$.
- *If there exists $z_m \in I$ such that $\Omega(z) > \Omega(z_m) > 0$ for all z in I different from z_m, then*: $\Delta z_k < \frac{\pi}{\sqrt{\Omega(z_m)}}$.
- *If $\Omega(z)$ changes sign at most once in I and it is strictly increasing when it is positive, then*: $\Delta^2 z_k < 0$.
- *If $\Omega(z)$ changes sign at most once in I and it is strictly decreasing when it is positive, then*: $\Delta^2 z_k > 0$.

We will apply the Sturm theorem after transforming the Gauss hypergeometric equation to normal form by means of LG transformations, which we discuss in the next section.

It should be emphasized that the Sturm theorem applies to any solution of the differential equation; therefore, the results we will obtain will be globally valid for any solution. The other two types of globality properties (validity for all the real zeros and for all n) is obtained by carefully choosing the changes of variable for the LG transformation. The bounds obtained from Sturm theorem will be optimal in the sense that they are obtained by finding the extrema of $\Omega(z)$.

2.1. LG transformations

We start from the hypergeometric equation

$$\begin{aligned}&w''(x) + B(x)w'(x) + A(x)w(x) = 0,\\ &B(x) = \frac{c-(a+b+1)x}{x(1-x)}, \quad A(x) = -\frac{ab}{x(1-x)},\end{aligned} \tag{5}$$

which has the hypergeometric function ${}_2\mathrm{F}_1(a, b; c; x)$ as one of its solutions. We will consider values of the parameters a, b and c such that oscillation in the interval $(0, 1)$ is possible [2]. Considering the Jacobi notation

$$a = -n, \quad b = n + \alpha + \beta + 1, \quad c = \alpha + 1, \tag{6}$$

corresponding to the standard notation for Jacobi polynomials

$$P_n^{(\alpha,\beta)}(1-2x) = \binom{n+\alpha}{n} {}_2\mathrm{F}_1(-n, n+\alpha+\beta+1; \alpha+1; x), \tag{7}$$

the necessary conditions for oscillation can be written as [1]

$$n > 0, \quad n+\alpha > 0, \quad n+\beta > 0, \quad n+\alpha+\beta > 0. \tag{8}$$

If one of these conditions is not met, then any solution of the differential equation has less than two zeros in (0, 1).

From now on, we use this notation and assume that these conditions are verified. We will also denote

$$L \equiv 2n+\alpha+\beta+1. \tag{9}$$

The results of the paper apply to any solution of the hypergeometric differential equation. In the particular case of Jacobi polynomials, the parameter n is a positive integer value (the degree of the polynomial).

From the oscillatory conditions (8) it is easy to check that

$$L^2-\alpha^2-\beta^2-1-2c_1\alpha-2c_2\beta-2c_3\alpha\beta > 0 \quad \text{for any } c_1, c_2, c_3 \in [-1, 1]. \tag{10}$$

This property will be frequently used in the sequel, particularly for $c_i = -1, 0, 1$.

For applying Theorem 1 we need to transform Eq. (5) to normal form (Eq. (4)). For this sake, we LG-transform the equation. This means that we consider a change of variable $z(x)$, $z'(x) > 0$ in (0, 1) and the functions

$$y(z(x)) = \sqrt{z'(x)} \exp\left(\frac{1}{2}\int^x B(t)\,dt\right) w(x), \tag{11}$$

where $w(x)$ is any solution of Eq. (5). The functions $y(z)$ satisfy an equation in normal form in the z variable (4), where

$$\Omega(z) = \dot{x}(z)\tilde{A}(x(z)) + \frac{1}{2}\{x, z\}, \quad \tilde{A}(x) = A(x) - \frac{B'(x)}{2} - \frac{B(x)^2}{4}, \tag{12}$$

and $\{x, z\}$ is the Schwarzian derivative of x with respect to z (see [4, p. 191]).

Obviously, the new functions $y(z(x))$ have the same zeros as $w(x)$ if $B(x)$ is continuous, as is the case of Gauss hypergeometric equation in the interval (0, 1). The resulting differential equation for $y(z)$ can then be used for extracting information on the zeros of the solutions of the initial equation.

Theorem 1 provides information on the spacing between zeros, for any oscillatory solution of the hypergeometric equation, provided we are able to determine the monotonicity properties of $\Omega(z)$ or, equivalently (because $z'(x) > 0$), the properties of

$$\Omega(x) \equiv \Omega(z(x)) = \frac{1}{z'(x)^2}\left(A(x) - \frac{B'(x)}{2} - \frac{B^2(x)}{4} + \frac{3z''(x)^2}{4z'(x)^2} - \frac{z'''(x)}{2z'(x)}\right). \tag{13}$$

The problem is to determine the changes of variable for which the analysis of the monotonicity properties of $\Omega(z)$ or $\Omega(x)$ is affordable and provides global information. This requirement drastically restricts the possible changes of variables that can be taken into account.

2.2. Admissible changes of variable

Let us first consider the trivial change $z(x) = x$. The associated LG transformation takes the hypergeometric equation to normal form in the original variable. In this case we have

$$\Omega(x) = \frac{1}{4}\left[\frac{L^2-\alpha^2-\beta^2+1}{x(1-x)} + \frac{1-\alpha^2}{x^2} + \frac{1-\beta^2}{(1-x)^2}\right]. \tag{14}$$

Then we can write $\Omega(x) = x^{-2}(1-x)^{-2}P(x)$, where $P(x)$ is a second degree polynomial. This seems a tractable function, however, when computing the derivative we get

$$\Omega'(x) = x^{-3}(1-x)^{-3}Q(x), \tag{15}$$

$Q(x)$ being a polynomial of third degree with coefficients depending on three parameters. Although information may be obtained for some parameter values, the problem is hard to solve in general; to begin with, we would need to solve a third degree equation depending on three parameters (n, α and β) and to distinguish the cases depending on the number of real roots. Furthermore, even for particular cases for which this analysis is feasible, it hardly provides global information for all the zeros.

As an illustration of this, let us consider the symmetrical case $|\alpha| = |\beta|$, then we have

$$\begin{aligned}
\Omega(x) &= -\frac{L^2-1}{4x^2(1-x)^2}\left[\left(x-\frac{1}{2}\right)^2 + \gamma - \frac{1}{4}\right], \quad \gamma = (\alpha^2-1)/(L^2-1),\\
\Omega'(x) &= -\frac{L^2-1}{4x^3(1-x)^3}(2x-1)\left[\left(x-\frac{1}{2}\right)^2 + 2\gamma - \frac{1}{4}\right].
\end{aligned} \tag{16}$$

In this simple case, the situation can be very varied depending on the values of the parameters α, β and n (or L). For instance, when $\alpha = \beta > 1$, Ω has a positive maximum in $(0, 1)$ and goes to $-\infty$ as $x \to 0^+, 1^-$. However, depending on the value of n, it may have three positive relative extrema (two maxima and one minimum) or only one. As we depart from this symmetrical case, the analysis becomes much more difficult because we have a third degree polynomial in the derivative and we can no longer factor out the trivial factor $x - \frac{1}{2}$.

It is necessary to further simplify the analysis in order to obtain truly global properties which hold for all the zeros in $(0, 1)$ and hold independently on n (or L).

The idea is to consider changes of variable such that $\Omega(x)$ has a first derivative of the type

$$\Omega'(x) = x^m(1-x)^r Q(x), \tag{17}$$

$Q(x)$ being a second degree polynomial. This suggests considering changes of variable

$$z'(x) = x^{p-1}(1-x)^{q-1}. \tag{18}$$

For this, the LG-transformation provides the following $\Omega(x)$:

$$\begin{aligned}
\Omega(x) = {}& \frac{1}{4}x^{-2p+2}(1-x)^{-2q+2}\\
&\times\left(\frac{L^2-\alpha^2-\beta^2+1-2(p-1)(q-1)}{x(1-x)} + \frac{p^2-\alpha^2}{x^2} + \frac{q^2-\beta^2}{(1-x)^2}\right),
\end{aligned} \tag{19}$$

which has derivative

$$\Omega'(x) = \tfrac{1}{4}x^{-2p-1}(1-x)^{-2q-1}P(x). \tag{20}$$

$P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ is a polynomial of degree 3 with coefficients depending on L, α, β, p and q. Solving $\Omega'(x) = 0$ will be equivalent to solving a quadratic equation when we can extract a factor x or $(1-x)$ or when $a_3 = 0$. We have that

$$a_3 = \tfrac{1}{2}(1-p-q)[L^2 - (1-p-q)^2], \tag{21}$$

$$P(0) = -\tfrac{1}{2}p(p^2 - \alpha^2), \tag{22}$$

$$P(1) = \tfrac{1}{2}q(q^2 - \beta^2). \tag{23}$$

Therefore, choosing $p+q=1$, $p=0$ or $q=0$ we have the desired simplification, independently of the parameters α, β and n. The analysis of the functions $\Omega(x)$ and $\Omega'(x)$ will reveal the ranges of values of the parameters for which global Sturm properties (valid for any solution, for all real zeros and for any n) are available.

There are other possible choices which lead to a second degree equation for the derivative, but the properties which are obtained are implicit with respect to the parameters in the sense that the definition of the change of variable for the LG transformation depends on one or several of the parameters n, α or β. In particular, for the choice $|L| = |1-p-q|$, $z(x)$ depends on n, α and β through L. For the cases $|p| = |\alpha|$ and $|q| = |\beta|$, the change of variable depends on $|\alpha|$ or $|\beta|$.

We will briefly outline the implicit properties but we will mainly focus on the set $p+q=1$ (which has Szegö's properties [5] and Grosjean's inequality [3] as particular cases), and $p=0$ (or $q=0$), which includes not only Grosjean's inequality but other logarithmic inequalities too [1]. The explicit properties apply for all values of n; the maximum and minimum values of $\Omega(z)$, Theorem 1, will depend on n, but not the change of variables, which will be also independent of α and β.

Note that from Eq. (19) it follows that interchanging p and q is equivalent to interchanging α and β and x and $1-x$. This symmetry property can be used to further reduce the study of the changes of variable to

(1) $p+q=1$, $p \leqslant \frac{1}{2}$ (or $q \leqslant \frac{1}{2}$);
(2) $q=0$ (or $p=0$).

Summarizing the procedure will be as follows. We will consider those changes of variable of the form (18) which convert the hypergeometric equation (5) to a LG-transformed equation $\ddot{y} + \Omega(z)y = 0$, where $\Omega(z(x))$ is given by Eq. (14). The values of p and q are chosen in such a way that solving $\Omega'(x) = 0$ is equivalent to solving a quadratic equation, which restricts the changes of variables to four families, two of them implicit. The fact that the replacements $p \leftrightarrow q$, $\alpha \leftrightarrow \beta$, $x \leftrightarrow 1-x$ leave $\Omega(x)$ invariant simplifies the analysis. For each of these families of changes of variable, we analyze the monotonicity properties of $\Omega(x)$ (which are the same as those of $\Omega(z)$ because $z'(x) > 0$). Then, Theorem 1 will be applied for obtaining Sturm properties in the z variable, particularly for the explicit changes of variable.

The Sturm properties will involve differences Δz_k (3), where $z_j = z(x_j)$, x_k and x_{k+1} are consecutive zeros of any solution of (5), $0 < x_k < x_{k+1} < 1$, and $z(x)$ is one of the admissible changes. Therefore, we will obtain bounds for the distances

$$\Delta z_k \equiv \Delta_k(p, q) \equiv \int_{x_k}^{x_{k+1}} t^{p-1}(1-t)^{q-1}\, dt, \tag{24}$$

as well as monotonicity properties in the k index. Because the differences $\Delta_k(p, q)$ vary continuously with respect to p and q we will obtain Sturm properties depending continuously on these parameters.

For certain values of p and q (0, $\frac{1}{2}$ and 1), the differences are expressible in terms of elementary functions [1]. In the general case, they can be written in terms of ${}_2\mathrm{F}_1$ hypergeometric functions through the incomplete Beta function $B_x(p, q)$. For instance, when $p > 0$ we can take

$$z(x) = \int_0^x t^{p-1}(1-t)^{q-1}\, dt = B_x(p, q) = \frac{x^p}{p}\, {}_2\mathrm{F}_1(1-q, p; p+1; x) \tag{25}$$

and $\Delta_k(p, q) = z(x_{k+1}) - z(x_k)$. The expression in terms of ${}_2\mathrm{F}_1$ can be used for all $p \neq 0, -1, -2, \ldots$. When $q > 0$ we can also take

$$z(x) = -\int_x^1 t^{p-1}(1-t)^{q-1}\, dt = -B_{1-x}(q, p). \tag{26}$$

For $p = 0$ or $q = 0$ the above changes of variable do not make sense but the limit $p \to 0$ can be taken in (24) (or the corresponding expression from Eq. (26)). The changes of variable show a $\log(x)$ term when $p = 0$ and a $\log(1-x)$ term when $q = 0$. For instance, for $p = 0$ we can take

$$z(x) = \log(x) + (1-q)x\, {}_3\mathrm{F}_2(1, 1, 2-q; 2, 2; x), \tag{27}$$

which explains why the cases $p = 0$ and $q = 0$ will be named logarithmic.

2.3. Szegö–Grosjean inequalities ($p + q = 1$)

The case $p + q = 1$ includes Szegö's inequalities [5] ($p = q = \frac{1}{2}$) and Grosjean's inequality [3] ($p = 0$) as particular cases. Grosjean's inequality was proven to hold not only for Legendre polynomials [1] but also for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $|\beta| \leqslant 1$ and, more generally, to any solution of the corresponding second order differential equation; naturally, when $|\alpha| \leqslant 1$ a similar inequality exists (corresponding to $q = 0$).

In order to obtain the Szegö–Grosjean inequalities, we have to determine the monotonicity properties of

$$\begin{aligned} \Omega(x) &= \tfrac{1}{4} x^{-2p}(1-x)^{2p-2} P(x), \\ P(x) &= \left[-L^2x^2 + (L^2 + \alpha^2 - \beta^2 + 1 - 2p)x + p^2 - \alpha^2\right], \end{aligned} \tag{28}$$

in $(0, 1)$.

Considering the oscillatory conditions (8), it is easy to show that $P(x)$ has always two different real zeros because the discriminant, as a function of p, is a parabola with a positive minimum value. The information on the location of these zeros, together with the behaviour of $\Omega(x)$ around $x = 0, 1$, serves to elucidate when the first two cases of Theorem 1 take place. Then, for instance, when both zeros are in $(0, 1)$, $\lim_{x\to 0^+} \Omega(x) < 0$, $\lim_{x\to 1^-} \Omega(x) < 0$ and $\Omega(x)$ has a positive maximum in $(0, 1)$; contrary, when there are no zeros in $(0, 1)$, $\lim_{x\to 0^+} \Omega(x) > 0$ and $\lim_{x\to 1^-} \Omega(x) > 0$, then $\Omega(x)$ has a positive minimum in $(0, 1)$.

The exact location of these extrema, the computation of the maximal or minimal values and the study of the monotonicity properties, become simpler by considering the change $x = t/(t+1)$, which takes the point $x = 1$ to $t = +\infty$ and preserves the monotonicity properties. Then, denoting

$$\Omega(t) = \Omega(x(t)), \tag{29}$$

we have

$$\begin{aligned}
&\Omega(t) = \tfrac{1}{4}\, t^{-2p}\left[((p-1)^2 - \beta^2)t^2 + \Lambda t + p^2 - \alpha^2\right],\\
&\Omega'(t) = -\tfrac{1}{2} t^{-2p-1}\left[(p-1)((p-1)^2-\beta^2)t^2 + (p-1/2)\Lambda t + p(p^2-\alpha^2)\right],\\
&\Lambda = L^2 - \alpha^2 - \beta^2 + p^2 + (p-1)^2 > 0.
\end{aligned} \tag{30}$$

From these expressions, only by considering the signs of the coefficients, it is easy to obtain the number of zeros of $\Omega(t)$ and $\Omega'(t)$ in $(0, +\infty)$ (and therefore $x \in (0, 1)$). Combining this information with the behaviour as $t \to 0^+, +\infty$, it is easy to identify the parameter values for which Sturm bounds (Theorem 3) or monotonicity properties (Theorem 2) take place, except for certain parameter regions which need further analysis (Theorem 5).

Theorem 2 (*Szegö–Grosjean monotonicity*). *Except when* $|\alpha| = |\beta| = p = \frac{1}{2}$, *the distances* $\Delta_k(p, 1-p)$ *satisfy the following properties*:

(1) *If* $p \leqslant \frac{1}{2}$ $|\alpha| \geqslant p$ *and* $|\beta| \leqslant 1-p$, *then* $\Delta_k(p, 1-p)$ *is strictly decreasing as function of* $k \in \mathbb{N}$.
(2) *If* $p \geqslant \frac{1}{2}$, $|\alpha| \leqslant p$ *and* $|\beta| \geqslant 1-p$, *then* $\Delta_k(p, 1-p)$ *is strictly increasing as function of* $k \in \mathbb{N}$.

When $|\alpha| = |\beta| = p = \frac{1}{2}$ *the distances* $\Delta_k(p, 1-p)$ *are constant as a function of* $k \in \mathbb{N}$.

Theorem 3 (*Generalized Szegö's bounds*). *The distances* $\Delta_k(p, 1-p)$ *can be bounded as follows and in the following open regions of the* (α, β)*-plane*:

(1) $\Delta_k(p, 1-p) < \pi/\sqrt{\Omega(x_m)}$ *if* $|\alpha| < p$ *and* $|\beta| < 1-p$. $\Omega(x_m)$ *is the minimum value of* Ω *in* $(0, 1)$.
(2) $\Delta_k(p, 1-p) > \pi/\sqrt{\Omega(x_M)}$ *if* $|\alpha| > p$ *and* $|\beta| > 1-p$. $\Omega(x_M)$ *is the maximum value of* Ω *in* $(0, 1)$.

When $p \neq \frac{1}{2}$, *the region of validity of each inequality includes the part of the boundary of the corresponding open region which is not included in Theorem* 2.

When $p = \frac{1}{2}$, *the inequalities can be continuously extended to all the boundaries*; *the minimum or maximum at the critical cases* ($|\alpha| = \frac{1}{2}$ *or* $|\beta| = \frac{1}{2}$) *is reached at* $x = 0$ *or* 1.

Fig. 1 shows the different regions for which the case of Theorems 2 and 3 take place.

The proof of these theorems is lengthy but immediate, particularly in the variable $t \in (0, +\infty)$ (Eq. (30)). For instance when $0 < p < 1$, $|p| < |\alpha|$ and $|p-1| < |\beta|$ simultaneously then $\Omega(t)$ verifies $\Omega(0^+) = \Omega(+\infty) = -\infty$. It is easy to check that $\Omega(t)$ has two real positive roots, and therefore it has a maximum where Ω is positive (which is the first case of Theorem 1). The rest of the cases in Theorem 3 can be proved in a similar way.

On the other hand, considering the sign of the derivative $\Omega'(t)$ the cases for which Ω is monotonic (Theorem 2) are easily obtained. For instance, when $0 < p < \frac{1}{2}$, $p \leqslant |\alpha|$ and $|p-1| \geqslant |\beta|$ all the coefficients of the derivative have the same sign and then Ω is increasing (and the third case applies). The rest of the cases also follow by elementary considerations.

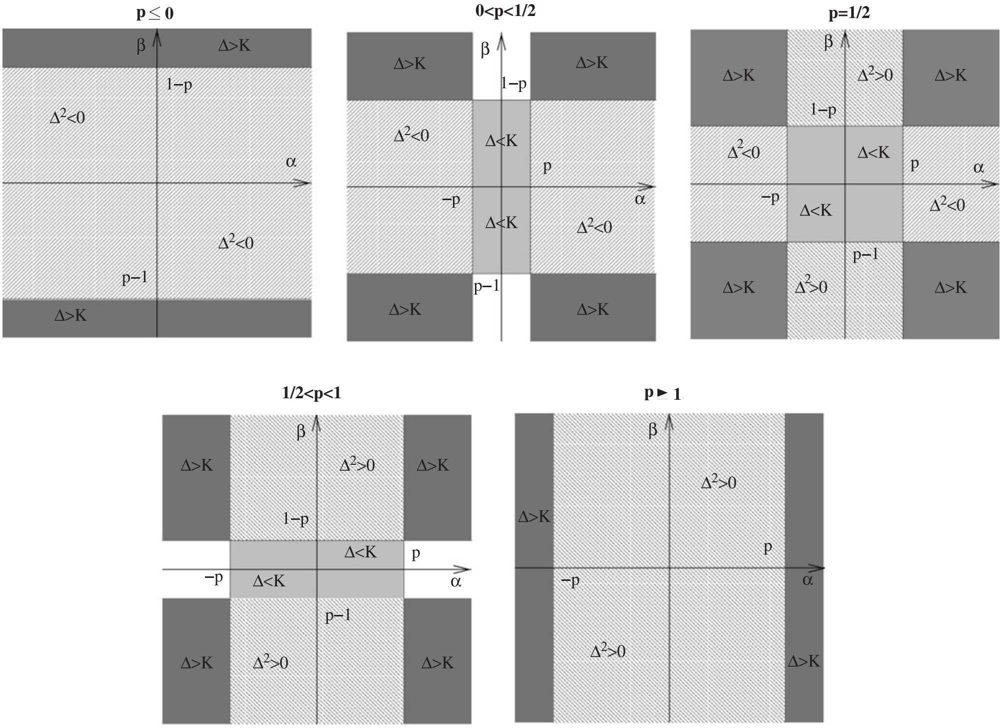

Fig. 1. Regions of the (α, β)-plane where different Sturm properties for the distances $\Delta_k(p, 1-p)$ are available. In the regions marked with the label "$\Delta < K$", $\Omega(x)$ has a minimum at certain $x = x_m$ and $\Delta_k(p, 1-p) < \pi/\sqrt{\Omega(x_m)}$. In the regions marked with the label "$\Delta > K$", $\Omega(x)$ has a maximum at certain $x = x_M$, and $\Delta_k(p, 1-p) > \pi/\sqrt{\Omega(x_M)}$. The label "$\Delta^2 > 0$" means that, in the corresponding regions, the distances $\Delta_k(p, 1-p)$ increase with k; when "$\Delta^2 < 0$" these distances decrease. In the white regions, no global Sturm properties are available for large n.

The cases already considered in [1], correspond to the first ($p = 0, q = 1$), third ($p = q = \frac{1}{2}$) and fifth pictures ($p = 1, q = 0$) of Fig. 1. The first and last pictures correspond to Grosjean's property (generalized) and the third picture corresponds to Szegö's properties (generalized). In the third picture (Szegö's), the four vertices $|\alpha| = |\beta| = \frac{1}{2}$ correspond to the Chebyshev cases.

2.3.1. *The Chebyshev cases*

Before computing the general bounds for the case $p + q = 1$, we will discuss the Chebyshev case in detail. Particular solutions for $|\alpha| = |\beta| = \frac{1}{2}$ when n is a positive integer are Chebyshev polynomials of the first ($\alpha = \beta = -\frac{1}{2}$), second ($\alpha = \beta = \frac{1}{2}$), third ($\alpha = -\frac{1}{2}, \beta = \frac{1}{2}$) and fourth ($\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$) kinds. For this parameter values equispacing of the zeros in the transformed variable $z(x) = \cos^{-1}(1 - 2x)$ takes place. These parameter values are very special cases, since no other values of p, q, α and β exist for which this property happens.

Theorem 4 (*Chebyshev cases*). *For the changes of variable with derivative* $z'(x) = x^{p-1}(1-x)^{q-1}$, *only when* $p = q = \frac{1}{2}$ *there exist cases for which the zeros in* $(0, 1)$ *are equally spaced in the* z *variable*; *these are the four cases with values* $|\alpha| = \frac{1}{2}$ *and* $|\beta| = \frac{1}{2}$, *which have Chebyshev* *polynomials as particular solutions. Furthermore, the case $\alpha = \beta = -\frac{1}{2}$ is the only one for which both the zeros and the extrema are equally spaced.*

For any solution of (5) *with parameter values* α, β *and* n (*Eq.* (6)) *consistent with oscillation* (*Eq.* (8)) *and such that* $|\alpha| = |\beta| = \frac{1}{2}$, *any two consecutive zeros* $x_k < x_{k+1}$ *in* (0, 1) *verify*

$$z(x_{k+1}) - z(x_k) = \frac{\pi}{n + (\alpha + \beta + 1)/2}. \tag{31}$$

If $\alpha = \beta = -\frac{1}{2}$ *and* $x'_k < x'_{k+1}$ *are two consecutive extrema in* (0, 1) *then*

$$z(x'_{k+1}) - z(x'_k) = \frac{\pi}{n}. \tag{32}$$

Notice that in the theorem, the restriction $p + q = 1$ is not considered; therefore, for proving it we should start from the general bi-parametric case (Eq. (19)). The theorem is easily proved by taking into account that for $\Omega(z(x))$ to be constant in (0, 1) it is necessary that $\Omega(z(x))$ is finite and positive as $x \to 0^+, 1^-$; then, none of the terms on the right-hand side of Eq. (19) can be unbound at these limits. For the last term this implies that $|q| = |\beta|$ or $q \leqslant 0$, but if $q < 0$ then $\Omega(z(x)) \to 0$ as $x \to 1^-$ and if $q = 0$ this limit is negative (the last term is dominant in this limit); therefore $|q| = |\beta|$. Using the same argument with the second term, we have $|p| = |\alpha|$ and then both the second and last terms must be zero. Then, when $|p| = |\alpha|$ and $|q| = |\beta|$ and considering (10) we have that the numerator of the first term is positive; the first term is then finite and positive as $x \to 0^+, 1^-$ only if $p = q = \frac{1}{2}$. Then only when $|p| = |q| = \alpha = \beta = \frac{1}{2}$ is $\Omega(z(x))$ constant.

On the other hand, the result for the extrema follows from the fact that, if y is a solution of the hypergeometric equation with parameter values a, b and c (α, β and n in the Jacobi notation), then the derivative y' is a solution of the hypergeometric equation with parameter values $a + 1$, $b + 1$, $c + 1$ ($n - 1$, $\alpha + 1$, $\beta + 1$). Therefore, the extrema for the case $\alpha = \beta = -\frac{1}{2}$ and some given n are zeros for the case $\alpha = \beta = \frac{1}{2}$, $n - 1$; these zeros are equispaced in the z variable with a distance given by Eq. (31), with n replaced by $n - 1$ and $\alpha = \beta = \frac{1}{2}$. For any other Chebyshev case different to $\alpha = \beta = -\frac{1}{2}$ equispacing of the extrema cannot occur for any solution, because the derivatives are not Chebyshev cases anymore.

2.3.2. *Computation of the bounds for* $p + q = 1$

The bounds in Theorem 3 are easily obtained by combining the equation $\Omega'(t) = 0$, $t > 0$, with the expression for $\Omega(t)$. Indeed, if $\Omega'(t_0) = 0$ then $(1 - p)((p - 1)^2 - \beta^2)t_0^2 = (p - \frac{1}{2})\lambda t_0 + p(p^2 - \alpha^2)$, and substituting in $\Omega(t)$ we have

$$\Omega(t_0) = \frac{1}{4}\, t_0^{-2p}(1 - p)^{-1}\left[\frac{1}{2}\,\Lambda t_0 + p^2 - \alpha^2\right]. \tag{33}$$

Similarly, we can write

$$\Omega(t_0) = \frac{1}{4}\, t_0^{-2(p-1)} p^{-1}\left[\frac{1}{2}\,\Lambda t_0^{-1} + (p - 1)^2 - \beta^2\right]. \tag{34}$$

Here t_0 is the only extrema for which $\Omega(t)$ is positive, namely

$$t_0 = \frac{-(p - 1/2)\Lambda \pm \sqrt{D_1}}{2(p - 1)[(p - 1)^2 - \beta^2]} = \frac{2p(p^2 - \alpha^2)}{-(p - 1/2)\Lambda \mp \sqrt{D_1}}, \tag{35}$$

where the upper sign is for the cases when $\Omega(t)$ has a positive maximum (Theorem 3(2)) and the lower sign when it is a minimum (Theorem 3(1)) and

$$D_1 = (p-1/2)^2\Lambda^2 - 4[(p-1/2)^2 - 1/4](p^2-\alpha^2)((p-1)^2-\beta^2). \tag{36}$$

If $p = \frac{1}{2}$, $t_0 = \sqrt{\dfrac{\alpha^2-1/4}{\beta^2-1/4}}$, independently of Λ (and n), and we obtain Szegö's bounds (in an improved and generalized version, see [1, Theorem 4]). If $p \neq \frac{1}{2}$ we have

$$t_0 = \frac{(1/2-p)\Lambda}{(\beta^2-(p-1)^2)(1-p)}(1+O(\Lambda^{-2})) \tag{37}$$

whenever $|\alpha| > p$, $|\beta| > 1-p$, $p \leqslant \frac{1}{2}$ (maximum) or $|\alpha| < p$, $|\beta| < 1-p$, $p \geqslant \frac{1}{2}$ (minimum). In this case, the extremum moves to the right as Λ increases. This means that the Sturm inequality tends to be sharper for the largest zeros as n increases.

On the other hand, we have

$$t_0 = \frac{p(\alpha^2-p^2)}{(p-1/2)\Lambda}(1+O(\Lambda^{-2})) \tag{38}$$

whenever $|\alpha| > p$, $|\beta| > 1-p$, $p \geqslant \frac{1}{2}$ (maximum) or $|\alpha| < p$, $|\beta| < 1-p$, $p \leqslant \frac{1}{2}$ (minimum), and the extremum tends to 0^+ as n increases.

Combining (35) with (33) and (34) we can write the bounds as follows:

$$\begin{aligned} \Delta_k(p,1-p) &> \begin{cases} K_1, |\alpha| > p, & |\beta| > 1-p, \quad p < 1/2, \\ K_2, |\alpha| > p, & |\beta| > 1-p, \quad p > 1/2, \end{cases} \\ \Delta_k(p,1-p) &< \begin{cases} K_2, |\alpha| < p, & |\beta| < 1-p, \quad p < 1/2, \\ K_1, |\alpha| < p, & |\beta| < 1-p, \quad p > 1/2, \end{cases} \end{aligned} \tag{39}$$

where

$$\begin{aligned} K_1 &= 2\sqrt{2}\pi\left[\frac{1/2-p}{\beta^2-(p-1)^2}\right]^{p-1/2}\left(\frac{\Lambda}{1-p}\right)^{p-1}(1+O(\Lambda^{-2})), \\ K_2 &= 2\sqrt{2}\pi\left[\frac{p-1/2}{\alpha^2-p^2}\right]^{1/2-p}\left(\frac{\Lambda}{p}\right)^{-p}(1+O(\Lambda^{-2})). \end{aligned} \tag{40}$$

This expression also gives the correct asymptotic behaviour for $p = \frac{1}{2}$ when $\Lambda \to \infty$, which can be seen by taking $p \to \frac{1}{2}$. Furthermore, when the bounds can be extended to the boundary of the open (α, β)-region where they are valid (for example $|\alpha| = p$, $0 < p \leqslant \frac{1}{2}$, see Theorem 3) the dominant term in the estimations are the exact bounds.

2.3.3. *Analysis of the remaining cases* ($p + q = 1$)

In the white regions in Fig. 1 neither Theorem 2 nor 3 apply. As we will now see, in this regions the function $\Omega(x)$ changes its behaviour as n increases in such a way that for large enough n no Sturm properties are available.

For these parameter ranges, it turns out that there are two possibilities for the behaviour of $\Omega(x)$ as $x \to 0^+, 1^-$:

(1) $\Omega(0^+) = +\infty, \Omega(1^-) = -\infty$ when $0 < p < \frac{1}{2}, |\alpha| < |p|, |\beta| > |1-p|$,
(2) $\Omega(0^+) = -\infty, \Omega(1^-) = +\infty$ when $\frac{1}{2} < p < 1, |\alpha| > |p|, |\beta| < |1-p|$.

Therefore, bounds for $\Delta_k(p, 1-p)$, as in Theorem 3, will not take place; but these distances could be monotonic as a function of k (increasing in the first case, decreasing in the second). However, there are parameter values for which two extrema exist (one maximum and one minimum) such that $\Omega(x)$ is positive at these extrema and therefore $\Delta_k(p, 1-p)$ is not monotonic.

From the signs of the coefficients of $\Omega'(t)$ one observes that the polynomial part of $\Omega'(t)$ either has two positive real roots or none; additionally, when two extrema appear, $\Omega(t)$ is positive at these extrema. Indeed, using Eq. (33) we observe that $\Omega(t_0)$, t_0 corresponding to any of the two extrema, is positive when $0 < p < \frac{1}{2}, |\alpha| < |p|$. When $\frac{1}{2} < p < 1$ we can proceed similarly using Eq. (34), which shows that $\Omega(t_0) > 0$ when $\frac{1}{2} < p < 1$ and $|\beta| < |p-1|$.

Therefore, we have monotonicity when the discriminant D_1 (Eq. (36)) of the polynomial part of $\Omega'(t)$ is such that $\Delta \leqslant 0$. As a function of Λ, the discriminant changes sign at

$$\Lambda_0 = \sqrt{\frac{4((p-1/2)^2 - 1/4)(p^2-\alpha^2)((p-1)^2-\beta^2)}{(p-1/2)^2}}, \tag{41}$$

being negative when $\Lambda < \Lambda_0$ and positive when $\Lambda > \Lambda_0$.

As a consequence, we have

Theorem 5 (*Restricted monotonicity*). *Let* $p \in (0, 1)$ *and* $(p^2 - \alpha^2)((p-1)^2 - \beta^2) < 0$. *Let* Λ *given by* (30) *and let* Λ_0 *be given by* (41).

Then, if $\Lambda \leqslant \Lambda_0$

(1) *If* $|p| > |\alpha|$ *and* $p < \frac{1}{2}$ *then* $\Delta_k(p, 1-p)$ *is increasing as a function of* k.
(2) *If* $|p| < |\alpha|$ *and* $p > \frac{1}{2}$ *then* $\Delta_k(p, 1-p)$ *is decreasing as a function of* k.

If $\Lambda > \Lambda_0$ *no global Sturm properties are available because* $\Omega(x)$ *has two positive local extrema and* $\Omega(0^+) = \pm\infty, \Omega(1^-) = \mp\infty$.

2.4. *Logarithmic inequalities* ($p = 0$ *or* $q = 0$)

In this case, the analysis is similar but slightly more involved. We provide some details of the analysis for $q = 0$. The analysis of the case $p = 0$ follows easily by symmetry: in each equation, we should interchange α and β, q and p and x and $\zeta = 1 - x$.

In terms of $t = x/(1-x)$ we have, when $q = 0$,

$$\begin{aligned} &\Omega(t) \equiv \Omega(x(t)) = \tfrac{1}{4} t^{-2p}(1+t)^{-2+2p} P(t), \quad P(t) = \left(-\beta^2 t^2 + \Theta t + p^2 - \alpha^2\right), \\ &\Theta = L^2 - \alpha^2 - \beta^2 - 1 + 2p. \end{aligned} \tag{42}$$

It can be checked (using (8)) that, as a function of p, the discriminant of the equation $P(t) = 0$ is a parabola with a positive minimum. Therefore, $P(t)$ has two real roots. Considering the signs of the coefficients, the number of positive roots of $\Omega(t)$ can be determined. Taking also into account the values of $\Omega(t)$ at 0^+ and $+\infty$ (and using that $\Theta > 0$ if $p \geqslant 0$ or $|p| \leqslant |\alpha|$ and (8) holds), the cases for which $\Omega(t)$ reaches a maximum where $\Omega(t)$ is positive can be identified. This takes place when $|\alpha| > p$, and also when $|\alpha| = p$ and $p \leqslant \frac{1}{2}$.

For locating the extrema and obtaining the value at the extremum, and also for studying monotonicity properties, one needs to compute the derivative of $\Omega(t)$ (or $\Omega(x)$). The analysis of these properties is slightly more involved in this case than in that of Szegö–Grosjean properties. First we summarize the corresponding results, including the case $p = 0$, and later on we give additional details regarding the computation of the bounds and the proof of the monotonicity properties.

Theorem 6 (*Logarithmic monotonicity*). *The distances $\Delta_k(p, q)$ verify the following*:

(1) $\Delta_k(p, 0)$ *is an strictly increasing function of $k \in \mathbb{N}$ when $p \geqslant \frac{1}{2}$, $|\alpha| \leqslant p$.*
(2) $\Delta_k(0, q)$ *is an strictly decreasing function of $k \in \mathbb{N}$ when $q \geqslant \frac{1}{2}$, $|\beta| \leqslant q$.*

Theorem 7 (*Logarithmic bounds*). *The distances $\Delta_k(p, q)$ for $p = 0$ or $q = 0$ verify*

(1) $\Delta_k(p, 0) > \pi/\sqrt{\Omega(x_M)}$ *if* $|\alpha| > p$.
(2) $\Delta_k(0, q) > \pi/\sqrt{\Omega(x_M)}$ *if* $|\beta| > q$.

$x_M \in (0, 1)$ *is the value for which $\Omega(x)$ is maximum.*

When $p, q \neq \frac{1}{2}$ the region of validity of each inequality includes the part of the boundary which is not included in Theorem 6.

If $|\alpha| = p = \frac{1}{2}$ or $|\beta| = q = \frac{1}{2}$ the bounds still hold but the maximum value of $\Omega(x)$ is reached at $x = 0$ or 1, *respectively.*

Theorem 8 (*Restricted monotonicity*). *The following restricted monotonicity properties hold*:

(1) *If $q = 0$, $0 < p < \frac{1}{2}$, $|\alpha| < p$, let*

$$F = L^2 + \alpha^2 - \beta^2 - p^2 - (p-1)^2$$

and

$$F_0 = \sqrt{\frac{4(p-1/2)^2 - 1}{(p-1/2)^2}((p-1)^2 - L^2)(p^2 - \alpha^2)}.$$

Then

(a) *If $F \leqslant F_0$, $\Delta_k(p, 0)$ is a strictly increasing function of k.*
(b) *If $F > F_0$, no global Sturm properties are available because $\Omega(x)$ has two positive local extrema and $\Omega(0^+) = +\infty$ while $\Omega(1^-) \leqslant 0$.*

(2) *If $p = 0$, $0 < q < \frac{1}{2}$, $|\beta| < q$, let*

$$F = L^2 - \alpha^2 + \beta^2 - q^2 - (q-1)^2$$

and

$$F_0 = \sqrt{\frac{4(q-1/2)^2 - 1}{(q-1/2)^2}((q-1)^2 - L^2)(q^2 - \beta^2)}$$

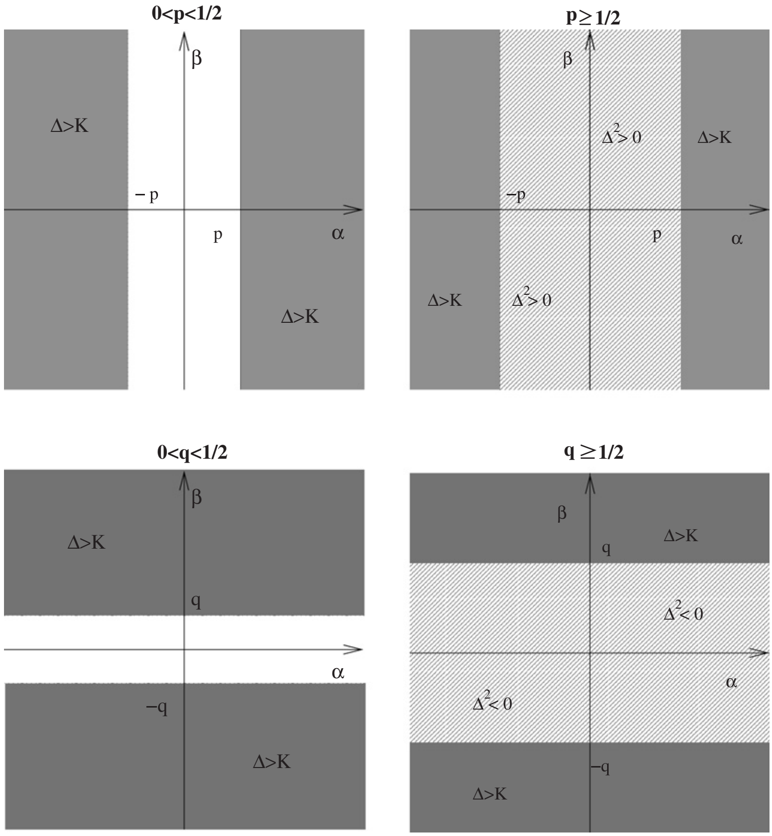

Fig. 2. Regions of the (α, β)-plane where different Sturm properties are available for the distances $\Delta_k(p, 0)$ (upper figures) and $\Delta_k(0, q)$ (lower figures). The meaning of the labels is as in Fig. 1.

then

(a) *If* $F \leqslant F_0$, $\Delta_k(0, q)$ *is a strictly decreasing function of* k.
(b) *If* $F > F_0$, *no global Sturm properties are available because* $\Omega(x)$ *has two positive local extrema and* $\Omega(0^+) \leqslant 0$ *while* $\Omega(1^-) = +\infty$.

Fig. 2 illustrates the different possibilities described in Theorems 6–8.

The proof of Theorem 6 was outlined before. With respect to the monotonicity properties and the restricted monotonicity theorem the proof can be made relatively simple by writing, when $q = 0$

$$\begin{aligned}
&\Omega(x) = \tfrac{1}{4}x^{-2p}P(x), \quad \Omega'(x) = -\tfrac{1}{2}x^{-2p-1}Q(x),\\
&P(x) = Ex^2 + Fx + G,\\
&Q(x) = (p-1)Ex^2 + (p-1/2)Fx + pG,\\
&E = (p-1)^2 - L^2, \quad G = (p^2 - \alpha^2), \quad F = -E - G - \beta^2
\end{aligned} \tag{43}$$

and the analogous relations if $p = 0$.

For proving monotonicity properties (restricted or not) in the case $q = 0$, which take place when $p > 0$, $|\alpha| \leqslant p$, it is useful to analyze the sign of the coefficients of $\Omega'(x)$. Because (as can be observed from (42)), $\Omega(x)$ has opposite sign as $x \to 0^+$ and $x \to 1^-$, then $\Omega(x)$ either has two extrema in (0, 1) or none. We can prove both monotonicity theorems by considering Vieta's formulas and the signs of the coefficients of $Q(x)$.

For the case of Theorem 6, $p \geqslant \frac{1}{2}$, the signs of the coefficients is such that there cannot be two positive solutions of $\Omega'(x) = 0$ when $p \leqslant L + 1$; when $p > L + 1$ there may be two positive roots, but there cannot be two roots in (0, 1), because, using Vieta's formulas, it is immediate to check that the roots are such that $x_1 x_2 > 1$. This proves the case $q = 0$ in Theorem 6. The case $p = 0$ is proven in a similar way.

When $0 < p < \frac{1}{2}$, Vieta's formulas for the coefficients of $Q(x)$, together with the oscillatory conditions, can be used to prove that the roots of the equation $Q(x) = 0$ verify $0 < x_1 + x_2 < 1$, $0 < x_1 x_2 < 1$. When the discriminant is positive there are two roots of $Q(x)$ in (0, 1); furthermore, $\Omega(x)$ is positive at these extrema (which can be checked using Eq. (44)), and therefore no Sturm property is available. Contrary, when the discriminant of the equation is negative there are no roots of $Q(x)$ and the monotonicity properties then apply. When the discriminant is exactly zero, there is a single root of $\Omega(x)$ and it is an inflexion point; the monotonicity properties also apply in this case.

For computing the bounds of Theorem 7, it is useful to combine the equation $\Omega'(x) = 0$, with $\Omega(x)$; with this, we have that if $\Omega'(x_0) = 0$, $x_0 \in (0, 1)$ then, for the case $q = 0$

$$\Omega(x_0) = \frac{1}{4} x_0^{-2p} (1 - p)^{-1} \left[\frac{1}{2} F x_0 + G \right]. \tag{44}$$

Solving the equation $\Omega(x) = 0$, and substituting into Eq. (44), we obtain the bounds

$$\Delta_k(p, 0) > K, \quad K = \pi / \sqrt{\Omega(x_0)}, \tag{45}$$

where x_0 is the maximum of $\Omega(x)$ in (0, 1) ($\Omega(x_0) > 0$).

A quite straightforward but rather lengthy analysis shows that the value x_0 always corresponds to the same root of the equation $Q(x) = 0$, namely

$$x_0 = \frac{-(p - 1/2)F + \sqrt{D_2}}{2(p - 1)E} = \frac{-2pG}{(p - 1/2)F + \sqrt{D_2}}. \tag{46}$$

D_2 is the discriminant of the equation, that is

$$D_2 = (p - 1/2)^2 F^2 - (4(p - 1/2)^2 - 1)EG. \tag{47}$$

For large L ($L \sim 2n$ when n is large, n corresponding to the degree in the polynomial case), we have the estimations

$$x_0 = \begin{cases} \dfrac{1/2 - p}{1 - p}(1 + O(L^{-2})), & p < 1/2, \\ \sqrt{\dfrac{\alpha^2 - 1/4}{L^2 - 1/4}}, & p = 1/2, \\ \dfrac{p(\alpha^2 - p^2)}{(p - 1/2)L^2}(1 + O(L^{-2})), & p > 1/2. \end{cases} \tag{48}$$

For large values of n, when $p < \frac{1}{2}$ the maximum is reached close to $x_0 \approx (\frac{1}{2} - p)/(1 - p)$, which approaches 0^+ as $p \to (1/2)^-$. For larger p, the maximum also tends to 0^+ as L becomes large. On the contrary, as p becomes a large negative number (for large L) the maximum tends to move toward $x = 1$. Of course, the bound is finer for zeros close to the maximum.

In the case $p = 0$, the results are the same interchanging α and β, p and q and x and $\zeta = 1 - x$. Here when $p \geqslant \frac{1}{2}$ the maximum approaches 1^- as $L \to +\infty$ while for negative q it tends to be closer to $x = 0$ as $|q|$ is larger.

With this estimations for the location of the maximum, the bounds in the case $q = 0$ (45) can be estimated as follows:

$$K = \begin{cases} 2\pi\sqrt{2(1-p)}\left(\dfrac{1-p}{1/2-p}\right)^{-p+1/2} L^{-1}(1+O(L^{-2})), & p < 1/2, \\ 2\pi\sqrt{2p}\left(\dfrac{p(\alpha^2-p^2)}{p-1/2}\right)^{p-1/2} L^{-2p}(1+O(L^{-2})), & p > 1/2. \end{cases} \tag{49}$$

The case $p = 0$ is analogous, with the usual replacements.

3. Implicit properties

The analysis of the two previous cases (Szegö–Grosjean properties and logarithmic inequalities) does not exhaust the possible Sturm properties which can be obtained through the analysis of the monotonicity properties of $\Omega(x)$.

There are additional selections of the parameters, namely $|p| = |\alpha|$ (or the similar case $|q| = |\beta|$) and $|L| = |1 - p - q|$, for which the analysis of Sturm properties is also possible. These properties are implicit in the sense that in the definition of $\Delta_k(p, q)$, p and/or q depend on one or several of the parameters n, α and β. We will not analyze these cases in detail, but only give the functions Ω and the derivative and describe some examples.

3.1. The cases $|p| = |\alpha|$ and $|q| = |\beta|$

In this section we describe Sturm properties for the distances $\Delta_k(\pm\alpha, q)$ ($|p| = |\alpha|$). The case $|q| = |\beta|$ is equivalent, with the changes of the parameters described before. These implicit properties are more easily described as a function of p and q, given fixed values of α and β.

As a function of $t = x/(1-x)$, when $|p| = |\alpha|$, the function $\Omega(t) = \Omega(x(t))$ and its derivative read:

$$\begin{aligned} &\Omega(t) = \tfrac{1}{4}t^{-2p+1}(1+t)^{2(p+q-1)}\left[(q^2-\beta^2)t + H\right], \\ &\Omega'(t) = \tfrac{1}{2}t^{-2p}(1+t)^{2p+2q-3}\left[q(q^2-\beta^2)t^2 + \zeta t + (1/2-p)H\right], \\ &H = L^2 - \alpha^2 - \beta^2 - 1 + 2(p+q-pq), \\ &\zeta = (1-p)(q^2-\beta^2) + (q-1/2)H. \end{aligned} \tag{50}$$

From the expression of $\Omega(t)$ and H, we observe that, for fixed L, $|\alpha|$, $|\beta|$ and p, H (and $\Omega(0^+)$) may be positive or negative depending on q. Indeed, at q equal to

$$q_c = \frac{L^2 - \alpha^2 - \beta^2 - 1 + 2p}{2(p-1)}$$

H changes sign. Therefore, the behaviour of $\Omega(t)$ changes at a q-value which depends on n, which complicates the analysis; the properties are then not global in the sense that they depend on n.

However, when restricting the study to values $|q| \leqslant |\beta|$, then $H > 0$ and the behaviour at $t = 0^+, +\infty$ (corresponding to $x = 0^+, 1^-$) becomes independent on n.

Let us briefly describe the case $|q| < |\beta|$; later we describe the case $|q| = |\beta|$. Only by considering the behaviour at $t = 0^+, +\infty$ it is possible to show that $\Omega(t)$ has a positive maximum when $p \leqslant \frac{1}{2}$, and therefore the first case of Theorem 1 applies and upper bounds for $\Delta_k(p, q)$ are available. On the other hand, when $|q| < |\beta|$, taking into account the values at $t = 0^+, +\infty$ and that the $\Omega'(t)$ as two positive roots at most, it is possible to show that the last case of Theorem 1 applies when $q < 0$, $p > \frac{1}{2}$, and therefore that $\Delta_k(p, q)$ is an increasing function of k.

All that remains, when $|q| < |\beta|$, is the case $q > 0$, $p \geqslant \frac{1}{2}$, where partial information can be obtained. When $0 < q \leqslant \frac{1}{2}$ and $\frac{1}{2} \leqslant p \leqslant 1$, considering the signs of the coefficients of $\Omega'(t)$ one sees that, again, $\Delta_k(p, q)$ is an increasing function of k.

In summary, $\Delta_k(p, q)$ when $|q| < |\beta|$ is an increasing function of k when

(1) $q < 0$ and $p \geqslant \frac{1}{2}$,
(2) $0 < q \leqslant \frac{1}{2}$, $\frac{1}{2} \leqslant p \leqslant 1$.

In the cases where $p \geqslant \frac{1}{2}$ not included in the previous two it is more difficult to decide, and there will be cases where the monotonicity property takes place and others where, like in Theorems 5 and 8, monotonicity may hold only for a restricted range of n-values. From the sign of ζ in Eq. (50), and taking into account the behaviour at $t = 0^+, +\infty$, we observe that as L becomes large (and then as n is large) $\Omega(t)$ becomes strictly decreasing when $0 < q < \frac{1}{2}$, whereas no Sturm properties are available when $q > \frac{1}{2}$ (because there are two extrema where $\Omega(t)$ is positive), similarly as in Theorems 5 and 8.

The case $|q| = |\beta|$, $|p| < |\alpha|$ can be analyzed similarly, with the corresponding replacements, yielding similar results (but with $\Omega(t)$ increasing when it is monotonic).

3.1.1. The case $|p| = |\alpha|$, $|q| = |\beta|$

When both $|p| = |\alpha|$ and $|q| = |\beta|$ the analysis is straightforward because the expression for $\Omega(x)$ is trivial, namely

$$\Omega(x) = \tfrac{1}{4}x^{-2p+1}(1-x)^{-2q+1}H, \tag{51}$$

with H as defined in Eq. (50).

Solving $\Omega'(x) = 0$, we see that the derivative is zero at

$$x_m = \frac{p - 1/2}{p + q - 1}. \tag{52}$$

When $p > \frac{1}{2}$, $q > \frac{1}{2}$ this corresponds to a minimum in (0, 1) and when $p < \frac{1}{2}$, $q < \frac{1}{2}$ to a maximum.

On the other hand, when $p < \frac{1}{2}$, $q > \frac{1}{2}$, $\Omega(x)$ is decreasing, while it is increasing when $p > \frac{1}{2}$, $q < \frac{1}{2}$.

Putting all this information together, also with the cases $p = \frac{1}{2}$ and/or $q = \frac{1}{2}$, we have the following two results.

Theorem 9 (*Sturm bounds for* $|p| = |\alpha|$, $|q| = |\beta|$). *Let* p, q *be such that* $|p| = |\alpha|$, $|q| = |\beta|$ *and let*

$$K(p,q) = \frac{2\pi}{\sqrt{H}}\left(\frac{p-1/2}{p+q-1}\right)^{p-1/2}\left(\frac{q-1/2}{p+q-1}\right)^{q-1/2} \tag{53}$$

with H *given by Eq.* (50), *then*

(1) *If* $p \geqslant \frac{1}{2}$, $q \geqslant \frac{1}{2}$ (*not both equal to* $\frac{1}{2}$) *then* $\Delta_k(p,q) < K(p,q)$.
(2) *If* $p \leqslant \frac{1}{2}$, $q \leqslant \frac{1}{2}$ (*not both equal to* $\frac{1}{2}$) *then* $\Delta_k(p,q) > K(p,q)$.
(3) *If* $p = \frac{1}{2}$, $q = \frac{1}{2}$ *then* $\Delta_k(p,q) = K(\frac{1}{2}, \frac{1}{2}) = 2\pi/\sqrt{H} = \pi/(n + (\alpha+\beta+1)/2)$ (*Chebyshev case*).

In the previous theorem, it is understood that $K(\frac{1}{2}, q)$, $K(p, \frac{1}{2})$, $K(\frac{1}{2}, \frac{1}{2})$ are the corresponding limits in Eq. (53).

Theorem 10 (*Monotonicity for* $|p| = |\alpha|$, $|q| = |\beta|$). *Let* p, q *be such that* $|p| = |\alpha|$, $|q| = |\beta|$, *then*

(1) *If* $p \geqslant \frac{1}{2}$, $q \leqslant \frac{1}{2}$ (*but not both* p *and* q *equal to* $\frac{1}{2}$), *then* $\Delta_k(p,q)$ *is strictly decreasing as a function of* $k \in \mathbb{N}$.
(2) *If* $p \leqslant \frac{1}{2}$, $q \geqslant \frac{1}{2}$ (*but not both* p *and* q *equal to* $\frac{1}{2}$), *then* $\Delta_k(p,q)$ *is strictly increasing as a function of* $k \in \mathbb{N}$.

3.2. The case $|L| = |1 - p - q|$

In this case, the functions $\Omega(t)$ and $\Omega'(t)$ are

$$\begin{aligned}
&\Omega(t) = \tfrac{1}{4} t^{-2p}(1+t)^{2p+2q-1}((q^2-\beta^2)t + (p^2-\alpha^2)),\\
&\Omega'(t) = \tfrac{1}{2} t^{-2p-1}(1+t)^{2(p+q-1)}\left[q(q^2-\beta^2)t^2 + \rho t - p(p^2-\alpha^2)\right],\\
&\rho = (\tfrac{1}{2} - p)(q^2-\beta^2) + (q - \tfrac{1}{2})(p^2-\alpha^2).
\end{aligned} \tag{54}$$

It is easy to check from Eq. (8) that $|L| = |1-p-q|$ implies that either $|p| > |\alpha|$ or $|q| > |\beta|$.

From Eq. (54), a number of properties can be obtained. The different values which make zero the coefficients of $\Omega(t)$ and $\Omega'(t)$ separate those regions where different properties take place. Particularly the values $|p| = |\alpha|$, $|q| = |\beta|$, $p = 0$ and $q = 0$ are important, and also $p = \frac{1}{2}$ and $q = \frac{1}{2}$ are relevant, given the expression for ρ.

The analysis would follow the lines of all previous analysis. However, the type of properties that can be obtained depend on n in the very same definition of $\Delta_k(p,q)$, because p and/or q depend on n through L. Besides, there is an implicit dependence on n in the expressions of $\Omega(t)$ and $\Omega'(t)$, which further complicates the analysis (particularly for analyzing the behaviour for large n). The properties are doubly implicit in n and very difficult to handle. We will not insist on analyzing this type of properties.

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