Qualitative and numerical aspects of the hypergeometric equation

PhD thesis

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Summary

In this thesis we present several analytical and numerical properties of functions which are solutions of Gauss, Kummer and Bessel differential equations, with real variable and parameters. These functions verify three different functional identities:

- Second order ordinary differential equations (ODE) in the variable x.
- Three term recurrence relations (TTRR) in the parameters.
- Systems of difference-differential equations (DDE), that relate pairs of hypergeometric functions and their derivatives.

This leads to the following three parts of this work:

1. Sturm-type properties of the real zeros of hypergeometric functions. Starting from the hypergeometric differential equations, written in the following form:

$$y''(x) + B(x)y'(x) + A(x)y(x) = 0, \tag{1}$$

Liouville-Green transformations are applied in order to derive an equation in normal form in a new variable z:

$$\ddot{Y}(z) + \Omega(z)Y(z) = 0. \tag{2}$$

The application of Sturm comparison theorem yields properties of the zeros of $Y(z)$, which are those of $y(x)$ up to the change of variable $z = z(x)$. If this change is simple enough, this information can then

be translated to the original variable x.

These Sturm-type properties follow from solving the problem $\dot{\Omega}(z)=0$ (or $\Omega'(x)=0$ if $z'(x)\neq 0$) and then analyzing the monotonicity properties of this function. In order to ensure that the problem is analytically affordable and that the resulting properties apply to all zeros within a given interval, we impose the following restriction [2]: solving $\dot{\Omega}(z)=0$ must be equivalent to solving an equation at most quadratic.

As a consequence of this restriction we obtain families of changes of variable $z=z(x)$ which are hypergeometric functions themselves, and that depend on some free parameters (two in the case of the Gauss equation and one in the case of the Kummer and Bessel equations). In [2] we analyze the different Sturm properties in terms of the relation between these parameters and those of the hypergeometric functions.

In [1] we consider several particular cases of these LG transformations, where the change of variable $z=z(x)$ can be written in terms of elementary functions. We recover some classical results of the theory of orthogonal polynomials (such as Szegö's bounds for the zeros of Jacobi polynomials), and we generalize other ones to a wider family of hypergeometric functions.

2. Numerical conditioning of hypergeometric three term recurrence relations. We analyze the numerical behaviour of hypergeometric TTRR:

$$y_{n+1}+b_n y_n+a_n y_{n-1}=0. \tag{3}$$

We recall the definitions of minimal and dominant solutions and Perron's theorem. It is well known that the computation of a minimal solution in a certain direction by means of the corresponding TTRR is an ill-conditioned problem. Conversely, Pincherle's theorem states that the associated continued fraction (CF):

$$\frac{y_n}{y_{n-1}}=\frac{-a_n}{b_n+}\,\frac{-a_{n+1}}{b_{n+1}+}\,\frac{-a_{n+2}}{b_{n+2}+}\cdots \tag{4}$$

converges if and only if the TTRR admits a minimal solution f_n, and it that case it converges to the ratio f_n/f_{n-1}.

In chapter 3 we note that the information obtained from Perron's theorem is of asymptotic nature, and that the behaviour of the TTRR for moderate values of n can be quite different to what is expected for large n. More precisely, we study the presence of transitory minimal solutions, which are solutions that are eventually dominant but behave temporarily like minimal ones [3]. We focus on the case where the coefficient a_n in (3) is negative and b_n changes sign once. The associated continued fractions show a temporal pseudoconvergence to a ratio of dominant solutions (which can be approximated with very high accuracy) before switching to the right limit.

We include the recursions for modified Bessel functions, confluent and Gauss hypergeometric functions as examples, as well as estimations of the numerical error in the computation of the dominant and minimal solutions when transitory minimal solutions are present.

3. Fixed point methods for the computation of real zeros of hypergeometric functions. We consider two pairs of functions $\{y^{(1)}(x), y^{(2)}(x)\}$ and $\{w^{(1)}(x), y^{(2)}(x)\}$, which are independent solutions of

$$y''(x) + A_y(x)y(x) = 0, \qquad w''(x) + A_w(x)w(x) = 0,$$

respectively. Typically, the functions $y^{(i)}(x)$ and $w^{(i)}(x)$ will be hypergeometric functions of the same family with different parameters. It can be shown that the pairs $\{y^{(1)}(x), w^{(1)}(x)\}$ and $\{y^{(2)}(x), w^{(2)}(x)\}$ satisfy a unique system of difference-differential equations (DDE):

$$\begin{aligned} y'(x) &= a(x)y(x) + d(x)w(x) \\ w'(x) &= b(x)w(x) + e(x)y(x) \end{aligned} \tag{5}$$

A change of variable $z(x) = \int^x \sqrt{-d(t)e(t)}dt$ and a normalization yield a transformed system in the variable z. From this system, a fixed point scheme can be constructed to compute the real zeros of $y(x)$. This method can be applied to different families of hypergeometric functions, namely Bessel functions, Coulomb wave functions and confluent and Gauss hypergeometric functions. The specific example of the Kummer function $M(a; c; x)$ is considered in [4], and the complete Maple package **zerosSF** for the computation of real zeros of hypergeometric functions will be developed using this algorithm [5].

This fixed point method relies on the evaluation of different ratios of hypergeometric functions, a task that can be carried out either by means of Maple internal subroutines or by using continued fractions. In the analysis the following points are considered: accuracy of the computation using the different continued fractions available (taking into account possible cases of pseudoconvergence) and comparison of CPU times between Maple and continued fraction evaluation.

References

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